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J. C. W. ROGERS

THE JOHNS HOPKINS UNIVERSITY ■ APPLIED PHYSICS LABORATORY

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In this report, which is the first part of a more comprehensive work, ~~we pose, and give~~ a solution algorithm for a hyperbolic conservation law. The algorithm is of an embedding type in that the solution is built up from Green's functions for simple processes taking place without reference to boundaries, and the locations of shocks are not explicitly followed. There is a discussion of boundary conditions and treatment of the Burgers and Korteweg-de Vries equations. Cursory mention is made of the extension to systems. Appropriate function spaces for solutions are introduced. The effects of perturbations in the initial conditions and of the velocity of propagation of disturbances are analyzed. For a monotonic velocity profile, in the one-dimensional case without boundaries, a convergence proof and error estimate are given.

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J. C. W. ROGERS

THE JOHNS HOPKINS UNIVERSITY ■ APPLIED PHYSICS LABORATORY
Johns Hopkins Road, Laurel, Maryland 20810
Operating under Contract N00017-72-C-4401 with the Department of the Navy

ABSTRACT

In this report, which is the first part of a more comprehensive work, we pose, and give a solution algorithm for, a hyperbolic conservation law. The algorithm is of an embedding type, in that the solution is built up from Green's functions for simple processes taking place without reference to boundaries, and the locations of shocks are not explicitly followed. There is a discussion of boundary conditions and treatment of the Burgers and Korteweg-de Vries equations. Cursory mention is made of the extension to systems. Appropriate function spaces for solutions are introduced. The effects of perturbations in the initial conditions and of the velocity of propagation of disturbances are analyzed. For a monotonic velocity profile, in the one-dimensional case without boundaries, a convergence proof and error estimate are given.

FOREWORD

This report and the ones to follow will comprise the bulk of a work entitled "Algorithms for Hyperbolic and Hydrodynamic Free Boundary Problems". I gratefully acknowledge the support for this work provided by the Office of Naval Research under Task No. NR 334-003. This task, with the title "Ship-Wave Interactions", has as its purpose the computation, in an efficient and reliable manner, of the phenomena attendant to the interaction of a rigid body with water possessing a free surface. It is also a pleasure for me to indicate the time I spent at the Applied Mathematics Research Institute, sponsored by the Office of Naval Research under Contract No. N00014-75-C-0921 with the Applied Institute of Mathematics, Inc., at Dartmouth College during the summers of 1975 and 1976, when parts of this work were done. It is my intention to have the whole work published separately under one cover once the various parts have been assembled.

The algorithms presented here are all semi-analytical, in that time, but not space, is discretized. Solutions to free boundary problems are built up out of solutions to simple linear partial differential equations. The particular type of quadrature scheme used to solve these simple equations approximately, and thereby effect the final numerical implementation of the algorithms on a computer, is left to the discretion of the reader. All proofs of convergence refer to the semi-analytical algorithms presented.

In the work to follow, there will be some unevenness of rigor and completeness, for which we offer no apology. In the first part of the work we present some algorithms for the solution of free boundary problems. The hyperbolic problems are not motivated so much physically as they are mathematically, to illustrate the ideas that we bring to bear on the hydrodynamic problem. There is more physical motivation for the algorithm for hydrodynamic flows, and we present the motivation with the algorithm. More recondite features of the algorithm, however, are treated more fully only at the end of this work.

In analyzing the well-posing of the hydrodynamic initial value problem in the sense of its unique and stable solvability by the computational procedure to be presented, we will need to clarify the various types of instabilities which can occur. Among the flows which can evolve in this way from the initial conditions are those which may conventionally be regarded as "turbulent". We will show the relationship between stability, energy conservation, and turbulence. Accordingly, a considerable portion of the work will contribute toward the outline of a rational theory of inviscid incompressible flow, especially the free boundary problem, and the

development of a preliminary theory of hydrodynamic turbulence. The essential physical ideas for achieving these objectives will be clarified, and the outstanding mathematical questions will be put into sharper focus.

I plan to follow this with a later work, which will contain a complete proof of the convergence of the algorithms presented here, the development of a full theory of turbulence for inviscid incompressible flows, and the proof of regularity results for general turbulent and non-turbulent flows.

CONTENTS

ILLUSTRATIONS

CHAPTER ONE

A HYPERBOLIC FREE BOUNDARY PROBLEM

1. Statement of the Problem and Presentation of a Solution Algorithm

We start by considering the initial value problem

$$u_t + v(u)u_x = 0 \ , \quad -\infty < x < \infty \ , \quad t > 0 \ , \tag{1.1.1a}$$

$$u(x,0) = u_0(x) \ , \quad -\infty < x < \infty \quad . \tag{1.1.1b}$$

This is a model hyperbolic initial value problem, and we will present a semi-analytical algorithm for its solution. The proof of convergence of the algorithm, for important classes of functions $v(\cdot)$, will be presented in Chapter Two.

As is well known, there is generally no guarantee that problem (1.1.1) possesses a unique solution, globally in time. Even if a unique solution exists for an initial time interval, there will be a tendency for internal discontinuities ("shocks") to develop. After the appearance of a "shock", equation (1.1.1) can be solved only in a generalized sense, and must be supplemented by various physically motivated "shock conditions" which are prescribed at each such discontinuity. Thus, the problem becomes a free boundary problem, since the equation (1.1.1) will at best possess classical solutions only throughout domains which do not contain shocks, but which have shocks as boundaries, and the locations of the shocks must be determined as part of the solution of the problem.

Since problem (1.1.1) is not in itself physically motivated, we will artificially provide an underlying "physical" principle by recasting it in "conservation" form:

$$u_t + (F(u))_x = 0 \ , \tag{1.1.2a}$$

$$F(u) = \int^u v(\xi)d\xi \quad . \tag{1.1.2b}$$

Assume that $\int_{-\infty}^{\infty} u_0 dx$ and $\int_{-\infty}^{\infty} u_0^2(x)dx$ are defined, and that $u(x) \to 0$

as $|x| \to \infty$. We get

$$U_1 \equiv \int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx \quad . \tag{1.1.3}$$

In developing an algorithm to solve (1.1.1), we will take care to conserve U_1 in (1.1.3). Moreover, we shall supplement the problem by the condition that

$$\frac{d}{dt} U_2(t) \leq 0 \ , \tag{1.1.4a}$$

where

$$U_2 \equiv \int_{-\infty}^{\infty} u^2(x,t)dx \ . \tag{1.1.4b}$$

(It should be pointed out that the conditions (1.1.3) and (1.1.4) serve only as mathematical motivation, so that we may illustrate, for the simple problem (1.1.1), ideas which will be developed more fully later. In fact, we will see in a later report that an equation like (1.1.1a), with $v(u) \equiv u$, arises in a very simple flow problem, but that the equation by itself is in the most general case devoid of meaning unless it is coupled to an equation $\rho_t + (\rho u)_x = 0$. In this case, the underlying physics of the problem shows that the correct generalized solution of (1.1.1a) depends critically on the function $\rho(x,0) \equiv \rho_0(x)$.)

The algorithm we use to solve (1.1.1), supplemented by (1.1.3) and (1.1.4), is of an "embedding" type, in which the shocks are not followed explicitly. Such a method is very efficient to treat cases where there are many shocks, or when there are several independent "space" variables x, as we shall see in the following. The technique has something in common with other embedding techniques like the method of artificial viscosity (Reference 11), except that here no extraneous parameter is introduced, beyond the time step τ in terms of which the evolutionary problem is solved. The method we use is built on the basic conservation "law" (1.1.3), which we regard as more fundamental than the differential equation (1.1.1a). Similar embedding methods based on conservation laws have been successfully used for other free boundary problems (References 2 and 3).

Before proceeding with an algorithm to solve (1.1.1), let us consider a slightly generalized version of the problem, where the solution of

$$u_t + (F(u))_x = 0, \quad x_0 < x < x_1 , \quad t > 0, \tag{1.1.5a}$$

$$u(x,0) = u_0(x), \quad x_0 \le x \le x_1 \ , \tag{1.1.5b}$$

supplemented by boundary conditions at x_0 and x_1, is sought. (It turns out that some natural boundary conditions for the problem are expressed simply in terms of a distribution function which appears in the execution of the solution algorithm. Therefore, an explicit statement of the boundary conditions is deferred until we describe the solution algorithm in detail, below.) We thus have an initial-boundary value problem.

Generalizations of equations (1.1.3) and (1.1.4) are

$$\frac{d}{dt}\int_a^b u(x,t)dx = -F(u(b,t)) + F(u(a,t)), \quad x_0 \le a < b \le x_1 \ , \tag{1.1.6}$$

and

$$\frac{d}{dt}\int_a^b u^2(x,t)dx \le -F_2(u(b,t)) + F_2(u(a,t)), \quad x_0 \le a < b \le x_1 \ , \tag{1.1.7a}$$

where

$$F_2(u) = 2\int^u \xi \ v(\xi)d\xi \ . \tag{1.1.7b}$$

Before we proceed with a precise description of our solution algorithm for the initial and boundary value problem associated with (1.1.5), let us give a schematic description. To facilitate this description, we shall assume that $u(x,t) \ge 0$ for all $x \in [x_0, x_1]$. We may assume we have a graph of the function $u(\cdot,t)$, for t fixed, as shown in Figure 1.

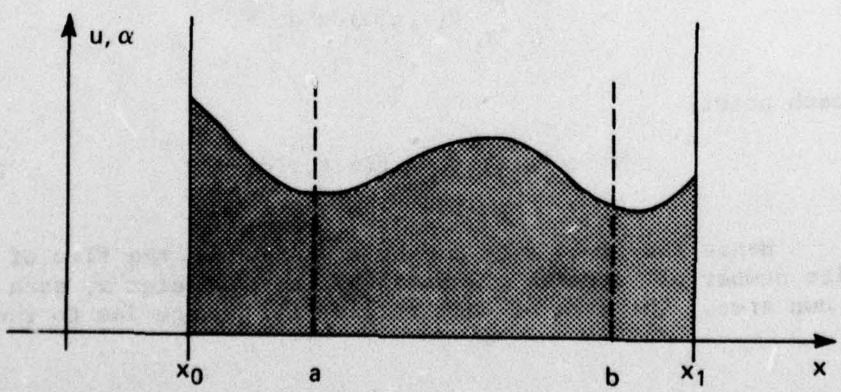

Figure 1. Schematic Solution u (·, t), for t Fixed.

Since in our conservation law (1.1.6) changes in the quantity $\int_a^b u(x,t)dx$ are brought about through terms which are functions only of u at the end points a and b, it is suggestive for us to think of $\int_a^b u\,dx$ as the total amount of some "fluid" which is conserved, except for the amount "flowing" across the vertical lines x = a and x = b. In order to avoid confusion of notation, we think of the plane in Figure 1 as the $x\alpha$-, instead of xu-, plane. The quantity $\int_a^b u(x,t)dx$ is just the area in the $x\alpha$-plane contained between the curves $\alpha = 0$ and $\alpha = u(x,t)$ and the vertical lines x = a and x = b. Thus, we may think of the flow as area-preserving, except for amounts crossing x = a and x = b. One of the principal features of our algorithm is that this area is conserved, except for such flow across the boundary.

In addition, we may think of the "fluid" as comprised of "streams", each moving along at constant α. The portion of fluid contained between α and $\alpha + d\alpha$ and the vertical lines x = a and x = b has the "area"

$$d\alpha \int_a^b f(x,t,\alpha)dx$$

where

$$f(x,t,\alpha) = \begin{cases} 1 & 0 \le \alpha \le u(x,t) \\ 0 & \alpha > u(x,t) \end{cases} \quad . \qquad (1.1.8)$$

The whole fluid area between x = a and x = b is

$$\int_0^\infty \int_a^b f(x,t,\alpha)dx\,d\alpha \quad .$$

At each point

$$u(x,t) = \int_0^\infty f(x,t,\alpha)d\alpha \quad . \qquad (1.1.9)$$

Hence the fluid flow may be envisaged as the flow of an infinite number of "streams", labeled by the parameter α, each with its own area. The area of each stream will change due to the flow

of fluid in the stream across the spatial boundaries, and also due to transfer of fluid from one stream to another. The flow of fluid in a stream across the spatial boundaries will be governed by the boundary conditions. The transfer of fluid from one stream to another will take place in such a way that the total area in the streams is conserved. The rules governing the manner in which fluid is transferred from one stream to another internally will constitute an expression of the shock conditions which are imposed. The various streams are indicated in Figure 1 by the horizontal lines.

To implement our solution algorithm we first convect each stream independently according to the linear equation

$$f_t(x,t,\alpha) + v(\alpha)\, f_x(x,t,\alpha) = 0 \qquad (1.1.10)$$

for a small time interval, called a time step; Equation (1.1.10) can be solved exactly. We sweep through the streams in the order of increasing α.

When $v(\alpha) > 0$ ($v(\alpha) < 0$) the stream between α and $\alpha + d\alpha$ enters (exits) the flow domain at x_0 and exits (enters) at x_1. As is well known, for equations like (1.1.10) we should thus specify $f(x,t,\alpha)$ at x_0 when $v(\alpha) > 0$ and at x_1 when $v(\alpha) < 0$. This information will be provided by the boundary conditions. When the range of values of u in the problem is such that we will have $v(u) > 0$ in one region and $v(u) < 0$ in another, we may generally expect that it will be necessary to prescribe some sort of boundary data at <u>both</u> x_0 and x_1, but in such a way as not to overdetermine the problem.

Figure 2 shows schematically the domain in the $x\alpha$-plane occupied by the fluid in Figure 1 after convection. The arrows point to the right for $v(\alpha) > 0$, to the left for $v(\alpha) < 0$.

If, after convection, the stream corresponding to $\alpha = \alpha_0$ has flowed into a spatial domain which has not been filled with all the streams for $\alpha < \alpha_0$, we let the "fluid" in the stream $\alpha = \alpha_0$ "fall" or "cascade" down to occupy the smallest unfilled values of $\alpha < \alpha_0$ at each point in the domain, in such a way that the total area of the streams remains unchanged. The result is shown in Figure 3.

A more mathematical description of our method for obtaining an approximate solution to the problem (1.1.5), subject to

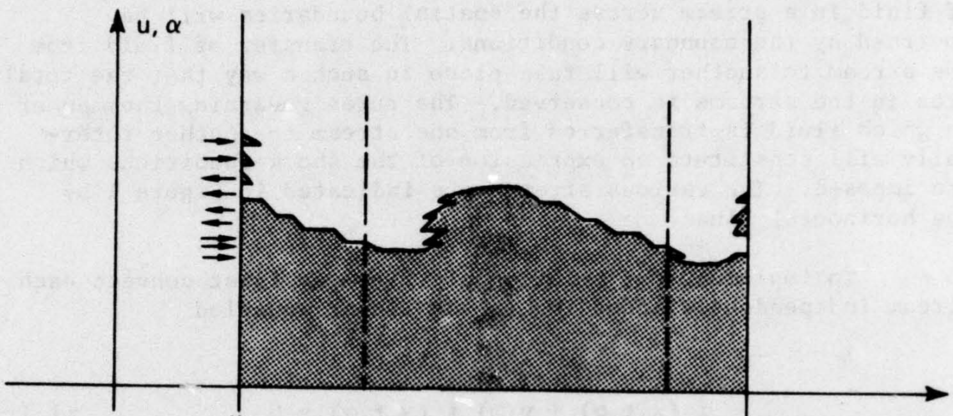

Figure 2. Streams in Figure 1 after Convection.

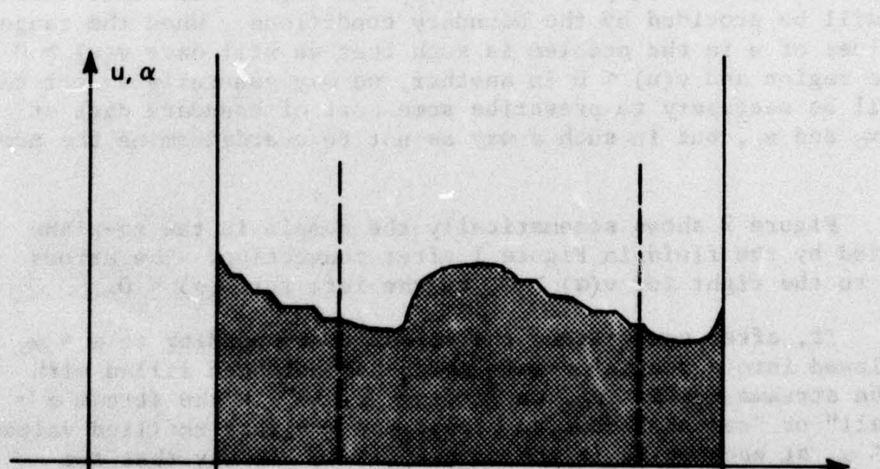

Figure 3. Streams in Figure 2 after Cascading.

boundary conditions to be specified, follows. We shall eliminate the restriction $u(x,t) \geq 0$ made in our heuristic description above. In addition to the initial function $u_0(x)$, we require functions $\tilde{u}(x_0,t)$ and $\tilde{u}(x_1,t)$ to be prescribed. These functions will be used to supply the boundary data required to solve Equation (1.1.10) uniquely, as described above. We assume that $u_0(x)$, $\tilde{u}(x_0,t)$, and $\tilde{u}(x_1,t)$ are bounded from below:

$$\begin{aligned} \tilde{u}(x_i,t) &\geq u_<, \quad i = 0,\ 1, \quad t \in (0,T] \ , \\ u_0(x) &\geq u_<, \quad x \in [x_0,x_1] \ , \end{aligned} \tag{1.1.11}$$

for some real number $u_<$. Let $\tau > 0$ be given. We start with

$$u^0(x) = u_0(x), \quad x \in [x_0,x_1] \ , \tag{1.1.12}$$

and the algorithm will be complete when we specify how to get from $u^n(x)$ to $u^{n+1}(x)$. Consider the solution of the linear problem

$$f^n_t\,(x,t,\alpha) + v(\alpha)\, f^n_x\,(x,t,\alpha) = 0 \tag{1.1.13a}$$

subject to the initial condition

$$f^n(x,0,\alpha) = \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \leq \alpha \leq u^n(x), \quad x \in [x_0,x_1] \\ 0 & \alpha > u^n(x) \end{cases} \tag{1.1.13b}$$

and the boundary conditions

$$f^n(x_0,t,\alpha) = \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \leq \alpha \leq \tilde{u}(x_0,t+n\tau), \quad v(\alpha) > 0, \quad 0 < t \ , \\ 0 & \alpha > \tilde{u}(x_0,t+n\tau), \quad v(\alpha) > 0 \end{cases} \tag{1.1.13c}$$

$$f^n(x_1,t,\alpha) = \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \le \alpha \le \tilde{u}(x_1,t+n\tau), \quad v(\alpha) < 0, \quad 0 < t. \\ 0 & \alpha > \tilde{u}(x_1,t+n\tau), \quad v(\alpha) < 0 \end{cases} \quad (1.1.13d)$$

Finally, let

$$u^{n+1}(x) = u_< + \int_{-\infty}^{\infty} f^n(x,\tau,\alpha)d\alpha \ , \qquad x \in [x_0,x_1] \ . \quad (1.1.14)$$

It is our hope that in some sense $u^n(x)$ will approximate the exact solution $u(x,n\tau)$ of problem (1.1.5), supplemented by suitable boundary conditions. We call the parameter τ the time step. In the next chapter we will prove the convergence of $u^n(x)$ to $u(x,t)$, as $\tau = t/n \downarrow 0$, or $n \uparrow \infty$, for important classes of problems of the sort (1.1.5), and we will obtain a bound on the error involved in approximating $u(x,t)$ by $u^n(x)$.

Note that the equation (1.1.13) for f^n may be solved explicitly. The solution is, for $0 \le t$,

$$f^n(x,t,\alpha) = \begin{cases} f^n(x,0,\alpha), & x \in [x_0,x_1], \quad \alpha \in C_0, \\ f^n\left(\max(x_0,x-v(\alpha)t),\ \max(0,t-\frac{x-x_0}{v(\alpha)}),\alpha\right), & x \in [x_0,x_1],\ \alpha \in C_+ , \\ f^n\left(\min(x_1,x-v(\alpha)t),\ \max(0,t-\frac{x-x_1}{v(\alpha)}),\alpha\right), & x \in [x_0,x_1],\ \alpha \in C_- \end{cases} \quad (1.1.15)$$

where

$$C_0 = \{\alpha|v(\alpha) = 0\}, \quad C_+ = \{\alpha|v(\alpha) > 0\}, \quad C_- = \{\alpha|v(\alpha) < 0\} . \quad (1.1.16)$$

Thus, at no step in the implementation of the algorithm need we take finite differences. The expression (1.1.15) may be thought of as giving the solution in terms of the initial and boundary data by means of Green's function for equation (1.1.13a), and Green's function for this problem is everywhere non-negative. The entire algorithm is executed as a series of integrations of non-negative quantities.

At this point let us state the boundary conditions we are imposing on the problem (1.1.5). We let the "density" $u(x,t)$ be

related to a "distribution function" $f(x,t,\alpha)$ by

$$u(x,t) \equiv If(x,t,\cdot) \equiv u_< + \int_{-\infty}^{\infty} f(x,t,\alpha)d\alpha, \quad x \in [x_0,x_1], \quad t \in [0,T] , \tag{1.1.17a}$$

$$f(x,t,\alpha) \equiv D(u(x,t);\alpha) \equiv \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \leq \alpha \leq u(x,t), \quad x \in [x_0,x_1],\ t \in [0,T] . \\ 0 & \alpha > u(x,t) \end{cases} \tag{1.1.17b}$$

For any $(x,t) \in [x_0,t_1] \times [0,T]$ we can decompose $u(x,t)$ as follows:

$$u(x,t) = \hat{u}(x,t) + u^+(x,t) + u^-(x,t) \tag{1.1.18}$$

where

$$\begin{aligned} \hat{u}(x,t) &= u_< + \int_{-\infty}^{\infty} \chi_0(\alpha)\ f(x,t,\alpha)d\alpha \\ u^+(x,t) &= \int_{-\infty}^{\infty} \chi_+(\alpha)\ f(x,t,\alpha)d\alpha , \\ u^-(x,t) &= \int_{-\infty}^{\infty} \chi_-(\alpha)\ f(x,t,\alpha)d\alpha , \end{aligned} \tag{1.1.19}$$

and

$$\chi_0(\alpha) = \begin{cases} 1 & v(\alpha) = 0 \\ 0 & v(\alpha) \neq 0 \end{cases} ,$$

$$\chi_+(\alpha) = \begin{cases} 1 & v(\alpha) > 0 \\ 0 & v(\alpha) \leq 0 \end{cases} , \tag{1.1.20}$$

$$\chi_-(\alpha) = \begin{cases} 1 & v(\alpha) < 0 \\ 0 & v(\alpha) \geq 0 \end{cases} .$$

Then the boundary conditions, which appear in the algorithm as (1.1.13c,d), in terms of the specified functions $\tilde{u}(x_0,t)$, $\tilde{u}(x_1,t)$,

when taken in conjunction with the initial conditions and the equation of evolution, amount to specifying $\lim_{x \downarrow x_0} u^+(x,t)$ and $\lim_{x \uparrow x_1} u^-(x,t)$. Only in special cases, such as when $v(\alpha) > 0$ for all α, so that $\chi_+(\alpha) \equiv 1$, do we find that this corresponds to prescribing u at one or the other of the end points.

A relation like (1.1.17a) may be used in conjunction with the expression (1.1.15) for $f^n(x,t,\alpha)$ to define a function $u^n(x,t)$:

$$u^n(x,t) = I\,(f^n(x,t-n\tau,\cdot)) \quad . \tag{1.1.21}$$

Note that, according to (1.1.15),(1.1.13), and (1.1.14),

$$\begin{aligned} u^n(x,n\tau) &= u^n(x)\ , && x \in [x_0,x_1]\ , \\ u^n(x,(n+1)\tau) &= u^{n+1}(x), && x \in [x_0,x_1]\ . \end{aligned} \tag{1.1.22}$$

With $u^n(x,t)$ defined in this manner, we can show that results similar to (1.1.6) and (1.1.7) are obtained, for $t-n\tau$ sufficiently small and suitable requirements of regularity on the initial function $u_0(x)$ and the boundary functions $\tilde{u}(x_i,t)$. To show this, we first note, from equation (1.1.15), that

$$\begin{aligned} \int_a^b f^n(x,t,\alpha)dx &= \int_a^b f^n(x,0,\alpha)dx + v(\alpha)\int_0^t f^n(a,t',\alpha)dt' \\ &\quad - v(\alpha)\int_0^t f^n(b,t',\alpha)dt' \quad . \end{aligned} \tag{1.1.23}$$

Integrate this over α and use (1.1.21), (1.1.13b), and (1.1.2b):

$$\begin{aligned} \int_a^b u^n(x,t+n\tau)dx &= \int_a^b u^n(x)dx + t\,F(u^n(a)) - t\,F(u^n(b)) \\ &\quad + \int_0^t \int_{-\infty}^{\infty} v(\alpha)\,[f^n(a,t',\alpha)-f^n(a,0,\alpha)-f^n(b,t',\alpha)+f^n(b,0,\alpha)]d\alpha\, dt' \ . \end{aligned} \tag{1.1.24}$$

Differentiating (1.1.24) with respect to t, we get

$$\frac{d}{dt}\int_a^b u^n(x,t+n\tau)dx = F(u^n(a)) - F(u^n(b))$$

$$+\int_{-\infty}^{\infty} v(\alpha)\ [f^n(a,t,\alpha)-f^n(a,0,\alpha)-f^n(b,t,\alpha)+f^n(b,0,\alpha)]d\alpha \tag{1.1.25}$$

In order to bound the last expression on the right of (1.1.25), we will find it convenient to introduce some more notation. Let

$$U \equiv \max\left(\sup_{x\in[x_0,x_1]} u_0(x),\ \sup_{t\in(0,T]} \tilde{u}(x_0,t),\ \sup_{t\in(0,T]} \tilde{u}(x_1,t)\right). \tag{1.1.26}$$

From equations (1.1.12) - (1.1.14) and (1.1.21), we see that

$$u^n(x,t) \le U \quad \text{and} \quad f^n(x,t,\alpha) = 0 \quad \text{for} \quad \alpha > U\ . \tag{1.1.27}$$

Further, define

$$V \equiv \sup_{\alpha\in[u_<,U]} |v(\alpha)| \quad . \tag{1.1.28}$$

Consider a representative contribution to the right hand side of (1.1.25):

$$\int_{-\infty}^{\infty} v(\alpha)\ [f^n(a,t,\alpha) - f^n(a,0,\alpha)]d\alpha \quad .$$

If $x_0 \le a - v(\alpha)t \le x_1$, write

$$f^n(a,t,\alpha) - f^n(a,0,\alpha) = f^n(a-v(\alpha)t,0,\alpha) - f^n(a,0,\alpha) \tag{1.1.29}$$

by (1.1.15), and note, from (1.1.13b) that the right hand side of (1.1.29) vanishes unless α lies between $u^n(a)$ and $u^n(a-v(\alpha)t)$. Thus, suppose $x_0 + Vt \le a \le x_1 - Vt$. It follows from (1.1.27) and (1.1.28) that $x_0 \le a - v(\alpha)t \le x_1$ for all α such that the difference in (1.1.29) is non-zero. We find

$$\int_{-\infty}^{\infty} |f^n(a,t,\alpha)-f^n(a,0,\alpha)|\, d\alpha \le \sup_{\xi_1,\xi_2\in[a-Vt,a+Vt]} (u^n(\xi_1)-u^n(\xi_2)) \tag{1.1.30}$$

in this case. If $a < x_0 + v(\alpha)t$, write

$$\begin{aligned} f^n(a,t,\alpha) - f^n(a,0,\alpha) &= f^n(x_0,0,\alpha) - f^n(a,0,\alpha) \\ &+ f^n\left(x_0, t-\frac{a-x_0}{v(\alpha)}, \alpha\right) - f^n(x_0,0,\alpha) \;. \end{aligned} \tag{1.1.31}$$

By the same reasoning that led to (1.1.30), we get, for $a < x_0 + Vt$ and $a \le x_1 - Vt$,

$$\begin{aligned} \int_{-\infty}^{\infty} |f^n(a,t,\alpha)-f^n(a,0,\alpha|\, d\alpha \le & \sup_{\xi_1,\xi_2\in[x_0,a+Vt]} (u^n(\xi_1)-u^n(\xi_2)) \\ &+\sup_{\eta\in[0,t]}(\tilde{u}^+(x_0,\eta+n\tau)-(u^n)^+(x_0)) + \sup_{\eta\in[0,t]}((u^n)^+(x_0)-\tilde{u}^+(x_0,\eta+n\tau)) , \end{aligned} \tag{1.1.32}$$

where $(u^n)^+$ and $\tilde{u}^+$ are obtained in terms of u^n and $\tilde{u}$ through relations like (1.1.17)-(1.1.20). The remaining contributions to the right hand side of (1.1.25) are bounded similarly, and we get

$$\left|\frac{d}{dt}\int_a^b u(x,t+n\tau)dx - F(u^n(a)) + F(u^n(b))\right| \le V\sigma_n(t) \tag{1.1.33a}$$

where

$$\sigma_n(t) = \sup_{\xi_1,\xi_2 \in [\max(x_0, a-Vt), \min(x_1, a+Vt)]} \left(u^n(\xi_1) - u^n(\xi_2)\right)$$

$$+ \sup_{\xi_1,\xi_2 \in [\max(x_0, b-Vt), \min(x_1, b+Vt)]} \left(u^n(\xi_1) - u^n(\xi_2)\right) \tag{1.1.33b}$$

$$+ \sup_{\eta \in [0,t]} \left(\tilde{u}^+(x_0, \eta+n\tau) - (u^n)^+(x_0)\right) + \sup_{\eta \in [0,t]} \left((u^n)^+(x_0) - \tilde{u}^+(x_0, \eta+n\tau)\right)$$

$$+ \sup_{\eta \in [0,t]} \left(\tilde{u}^-(x_1, \eta+n\tau) - (u^n)^-(x_1)\right) + \sup_{\eta \in [0,t]} \left((u^n)^-(x_1) - \tilde{u}^-(x_1, \eta+n\tau)\right) .$$

An integrated version of this equation becomes, on use of (1.1.22),

$$\left| \int_a^b u^n(x)dx - \int_a^b u_0(x)dx - \tau \sum_{i=0}^{n-1} [F(u^i(a)) - F(u^i(b))] \right| \le V\tau \sum_{i=0}^{n-1} \sigma_i(\tau) . \tag{1.1.34}$$

The right hand side of (1.1.33) will go to zero as $\tau \downarrow 0$ for $0 < t \le \tau$, if we have

(i) the u^n are continuous in some neighborhoods of a and b;

(ii) the boundary functions $\tilde{u}^+(x_0,\cdot)$ and $\tilde{u}^-(x_1,\cdot)$ are continuous in a neighborhood of $n\tau + t$;

(iii) $(u^n)^+(x_0) = \lim_{t \downarrow n\tau} \tilde{u}^+(x_0,t)$ and $(u^n)^-(x_1) = \lim_{t \downarrow n\tau} \tilde{u}^-(x_1,t)$.

Those regularity properties which relate to the calculated functions u^n will often follow from an a priori or a posteriori analysis of the problem. For example, in the important special case $v(\alpha) > 0$, $\alpha \in [u_<, U]$, we have (iii) above for $n > 0$. In the case $a = x_0 \to -\infty$ and $b = x_1 \to +\infty$, for most types of asymptotic conditions of interest, (i) holds. There will be mention of generalized versions of (1.1.33), for different classes of initial and boundary data, in Chapter Two.

A result similar to (1.1.7) follows if we multiply (1.1.23) by α and integrate over α. Note that

$$\int_{-\infty}^{\infty} \alpha\ f^n(x,0,\alpha)d\alpha = \frac{1}{2}\left[\left(u^n(x)\right)^2 - u_<^2\right] \tag{1.1.35a}$$

and

$$\int_{-\infty}^{\infty} \alpha\ f^n(x,t,\alpha)d\alpha \geq \frac{1}{2}\left[\left(n^u(x,t+n\tau)\right)^2 - u_<^2\right] . \tag{1.1.35b}$$

Thus

$$\int_a^b \left(u^n(x,t+n\tau)\right)^2 dx - \int_a^b \left(u^n(x)\right)^2 dx - tF_2\left(u^n(a)\right) + tF_2\left(u^n(b)\right) \leq G \tag{1.1.36a}$$

where

$$G \equiv 2\int_0^t \int_{-\infty}^{\infty} \alpha\ v(\alpha)\ [f^n(a,t',\alpha) - f^n(a,0,\alpha) \tag{1.1.36b}$$

$$- f^n(b,t',\alpha) + f^n(b,0,\alpha)]\, d\alpha dt' .$$

Proceeding as above, we get

$$|G| \leq 2t\left(|U| + |u_<|\right) V\sigma_n(t) . \tag{1.1.37}$$

The analogue to (1.1.34) is obtained by using (1.1.22) and (1.1.37):

$$\int_a^b \left(u^n(x)\right)^2 dx - \int_a^b \left(u_0(x)\right)^2 dx - \tau \sum_{i=0}^{n-1}\left[F_2\left(u^i(a)\right) - F_2\left(u^i(b)\right)\right] \leq H \tag{1.1.38a}$$

where

$$|H| \leq 2\tau\left(|U| + |u_<|\right) V \sum_{i=0}^{n-1} \sigma_i(\tau) . \tag{1.1.38b}$$

The right hand side of (1.1.37) will be $o(t)$ as $t \downarrow 0$, under conditions (i) - (iii) given above.

In view of the relations (1.1.33), (1.1.34), (1.1.36), (1.1.37), and (1.1.38), we may think of the algorithm (1.1.12)-(1.1.14) as "conserving" the quantity U_1 given in (1.1.3), and "diminishing" U_2 given in (1.1.4b), when those quantities are defined. It is natural, then, since U_1 and U_2 are given as integrals, to introduce the following set functions for Lebesgue-measurable sets Ω in $x\alpha$ space:

$$U_1(f;\Omega) \equiv \int_\Omega f(x,\alpha)\, d\alpha\, dx \quad , \tag{1.1.39a}$$

$$U_2(f;\Omega) \equiv 2\int_\Omega \alpha f(x,\alpha) d\alpha\, dx \quad , \tag{1.1.39b}$$

(Note that these definitions depend on the value of the number $u_<$ given in (1.1.11).)

We should point out that, if

(i) $u_0(x)$ is continuous for $x \in [x_0, x_1]$,

(ii) $\tilde{u}(x_0,t)$ is continuous for $t \in (0,\xi_0]$ and $\lim_{t\downarrow 0} \tilde{u}(x_0,t) = u_0(x_0)$,

(iii) $\tilde{u}(x_1,t)$ is continuous for $t \in (0,\xi_0]$ and $\lim_{t\downarrow 0} \tilde{u}(x_1,t) = u_0(x_1)$,

(iv) $x + v(u_0(x))\xi$ is nondecreasing in $x \in [x_0,x_1]$ for $0 \le \xi \le \xi_0$, (1.1.40)

(v) $\max[v(\tilde{u}(x_0,t))\,(\xi-t),0]$ is nonincreasing in $t \in [0,\xi_0]$ for $t \le \xi \le \xi_0$,

(vi) $\min[v(\tilde{u}(x_1,t))(\xi-t),0]$ is nondecreasing in $t \in [0,\xi_0]$ for $t \le \xi \le \xi_0$,

then for $0 \le n\tau \le \xi_0$, the quantity $u^n(x)$ computed by (1.1.12)-(1.1.14) will be the <u>exact</u> solution at time $t = n\tau$ of (1.1.5) with the boundary conditions $u^+(x_0,t) = \tilde{u}^+(x_0,t)$ and $u^-(x_1,t) = \tilde{u}^-(x_1,t)$. The quantity $u^n(x,t)$ given in (1.1.21) will be the exact solution at time t if $t \le \xi_0$.

To satisfy ourselves as to the validity of this assertion, we shall outline a proof for the case when $t = \tau \le \xi_0$ and $x_0 \to -\infty$, $x_1 \to +\infty$, so that we only need conditions (i) and (iv) above.

The exact solution of (1.1.1) at time $t = \tau$ is given by

$$u(x,\tau) = \alpha(x) \equiv \sup\{\alpha \mid \alpha - u_0(x - v(\alpha)\tau) \le 0\}. \tag{1.1.41}$$

To show that $u^1(x)$ given by (1.1.12)-(1.1.14) is $\alpha(x)$, we will make use of the following lemma.

Lemma 1.1.1: If $\alpha \ge u_<$, $f^o(y_1,0,\alpha) = 0$, and $f^o(y_2,0,\alpha) = 1$, then there exists $y^* \in [\min(y_1,y_2),\max(y_1,y_2)]$ such that $\alpha = u_0(y^*)$. Also, $y^* \ne y_1$.

Proof: The result follows from (1.1.12), (1.1.13b), and the continuity of u_0, since $\alpha > u_0(y_1)$ and $\alpha \le u_0(y_2)$.

To show that $u^1(x)$ is $\alpha(x)$, we show first that $f^o(x-v(\alpha)\tau,0,\alpha) = 1$ for $u_< \le \alpha < \alpha(x)$. We do this by contradiction. Suppose there is α_0, $u_< \le \alpha_0 < \alpha(x)$, with $f^o(x-v(\alpha_0)\tau,0,\alpha_0) = 0$. Then $\alpha_0 > u_0(x-v(\alpha_0)\tau)$. By the definition of $\alpha(x)$, equation (1.1.41), there is α_1, $\alpha_0 < \alpha_1 < \alpha(x)$, with $\alpha_1 - u_0(x-v(\alpha_1)\tau) \le 0$. We have the following relations immediately:

$$\begin{aligned} f^o(x-v(\alpha_0)\tau,0,\alpha_0) &= f^o(x-v(\alpha_0)\tau,0,\alpha_1) = 0 \;, \\ f^o(x-v(\alpha_1)\tau,0,\alpha_1) &= f^o(x-v(\alpha_1)\tau,0,\alpha_0) = 1 \;. \end{aligned} \tag{1.1.42}$$

We cannot have $v(\alpha_0) = v(\alpha_1)$ without violating the continuity of u_0. Consider the case $v(\alpha_0) > v(\alpha_1)$. Then by the lemma and (1.1.42) $\exists\, y_1^*$ such that

$$x - v(\alpha_0)\tau < y_1^* \le x - v(\alpha_1)\tau \tag{1.1.43}$$

and

$$u_0(y_1^*) = \alpha_1 \;. \tag{1.1.44}$$

Since $u_o(y_1^*) = \alpha_1$, $f^o(y_1^*,0,\alpha_1) = 1$ and

$$f^o(y_1^*,0,\alpha_o) = 1 \quad . \tag{1.1.45}$$

By the lemma, (1.1.42), and (1.1.45), $\exists\ y_o^*$ with

$$x - v(\alpha_o)\tau < y_o^* \le y_1^* \tag{1.1.46}$$

and

$$u_o(y_o^*) = \alpha_o \quad . \tag{1.1.47}$$

By (1.1.43), (1.1.44), (1.1.46), and (1.1.47),

$$y_o^* + v(u_o(y_o^*))\tau > x \ge y_1^* + v(u_o(y_1^*))\tau \ ,$$

which, with $y_o^* \le y_1^*$, violates condition (iv) of (1.1.40). The case $v(\alpha_o) < v(\alpha_1)$ is treated similarly. Thus, $f^o(x-v(\alpha)\tau,0,\alpha) = 1$ for $u_< \le \alpha < \alpha(x)$. Finally, suppose $f^o(x-v(\alpha)\tau,0,\alpha) = 1$ for some $\alpha > \alpha(x)$. Then $u_< \le \alpha \le u_o(x-v(\alpha)\tau)$ and by the definition of $\alpha(x)$, $\alpha \le \alpha(x)$, giving a contradiction. Putting these results together, we see that $u^1(x) = \alpha(x)$ follows from (1.1.14).

One may think of (1.1.13a) as a "Boltzmann equation", because of the analogy to the familiar transport equation of kinetic theory. The equation we consider is especially simple, and does not contain the nonlinear "collision terms" which generally appear in the Boltzmann equation. To say that our algorithm thus represents the motion of a "collisionless" fluid, however, would be inaccurate. Indeed, the transformation which takes place at each time step, when the "distribution function" $f^n(x,\tau,\alpha)$ is converted into $f^{n+1}(x,0,\alpha)$ by the processes of equations (1.1.14) and (1.1.13b), represents collisions of a rather extreme form. If we are to regard the transport equation (1.1.13a) as expressing the flow of independent "streams" at different "density levels" α, we see that at each time step, by some undefined internal "interaction" mechanism, the various streams interact in such a way that they are forced into the lowest "unfilled" density levels (above $u_<$). Accordingly, the algorithm (1.1.12)-(1.1.14) gives the evolution of the system in terms of a sequence of propagations of non-interacting streams, and intense interactions of these streams.

The fact that equation (1.1.5) for the density u may be related to an underlying transport equation for the distribution function f is not surprising, if one recalls that the equations of fluid mechanics may in some sense be derived from the Boltzmann equation (Reference 5).

2. Higher-Dimensional Cases

A higher-dimensional analogue of (1.1.5) is

$$u_t + \nabla \cdot (F(u)) = 0, \qquad x \in \mathfrak{D}, \qquad t > 0, \qquad (1.2.1a)$$

$$u(x,0) = u_0(x), \qquad x \in \bar{\mathfrak{D}}, \qquad (1.2.1b)$$

with boundary conditions to be specified. By $\bar{\mathfrak{D}}$ we mean the closure of $\mathfrak{D}$. Here F(u) is a vector. Its derivative with respect to u is another vector:

$$v(u) = \frac{d}{du} F(u) \quad . \qquad (1.2.2)$$

As to boundary conditions, let functions $\tilde{u}(x,t)$ be given for $x \in \partial\mathfrak{D}$ and $t \in (0,T]$, with

$$\begin{aligned} \tilde{u}(x,t) &\geq u_<, \qquad x \in \partial\mathfrak{D}, \qquad t \in (0,T] \quad , \\ u_0(x) &\geq u_<, \qquad x \in \bar{\mathfrak{D}} \quad . \end{aligned} \qquad (1.2.3)$$

If we write

$$u(x,t) \equiv If(x,t,\cdot) \equiv u_< + \int_{\infty}^{\infty} f(x,t,\alpha)d\alpha, \quad x \in \mathfrak{D}, \; t \in [0,T], \qquad (1.2.4)$$

$$f(x,t,\alpha) \equiv D(u(x,t);\alpha) \equiv \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \leq \alpha \leq u(x,t), \\ 0 & \alpha > u(x,t) \end{cases} \quad x \in \bar{\mathfrak{D}}, \; t \in [0,T], \qquad (1.2.5)$$

and define

$$C(x,t) \equiv \{\alpha \,|\, x - v(\alpha)\xi \in \bar{\mathfrak{D}} \text{ for all } \xi \in [0,t]\}, \quad x \in \bar{\mathfrak{D}} ,$$

$$C'(x,t) \equiv \{\alpha \,|\, \alpha \notin C(x,t)\}, \quad x \in \bar{\mathfrak{D}} , \tag{1.2.6}$$

we can decompose u(x,t) as

$$u(x,t) = \hat{u}(x,t,\xi) + u'(x,t,\xi) \tag{1.2.7}$$

where

$$\hat{u}(x,t,\xi) = u_< + \int_{-\infty}^{\infty} \chi(x,\xi,\alpha)\, f(x,t,\alpha)\,d\alpha ,$$

$$u'(x,t,\xi) = \int_{-\infty}^{\infty} \chi'(x,\xi,\alpha)\, f(x,t,\alpha)\,d\alpha , \tag{1.2.8}$$

and

$$\chi(x,\xi,\alpha) = \begin{cases} 1 & \alpha \in C(x,\xi) \\ 0 & \alpha \notin C(x,\xi) \end{cases} ,$$

$$\chi'(x,\xi,\alpha) = 1 - \chi(x,\xi,\alpha) . \tag{1.2.9}$$

For $t \in (0,T]$, $x \in \partial\mathfrak{D}$, and $\alpha \in C'(x,\xi)$ for all $\xi > 0$, we set

$$f(x,t,\alpha) = D(u(x,t);\alpha) . \tag{1.2.10}$$

The boundary conditions (1.2.10), when combined with (1.2.1), amount to specifying

$$\inf_{\xi>0} \lim_{y\in\mathfrak{D}\to x} u'(y,t,\xi), \quad x \in \partial\mathfrak{D}, \quad t \in (0,T] . \tag{1.2.11}$$

To solve (1.2.1) subject to the boundary conditions (1.2.11), we set

$$u^0(x) = u_0(x), \quad x \in \bar{\mathfrak{D}}, \tag{1.2.12}$$

and show how to get from $u^n(x)$ to $u^{n+1}(x)$. Given $u^n(x)$, construct

$$f^n(x,0,\alpha) = D(u^n(x);\alpha), \quad x \in \bar{\mathfrak{D}} \tag{1.2.13a}$$

and solve

$$f^n_t + v(\alpha) \cdot \nabla f^n(x,t,\alpha) = 0 \tag{1.2.13b}$$

subject to the initial condition (1.2.13a) and the boundary condition

$$f^n(x,t,\alpha) = f(x,t+n\tau,\alpha), \quad x \in \partial\mathfrak{D}, \quad t > 0, \quad \alpha \in C'(x,\xi) \text{ for all } \xi > 0, \tag{1.2.13c}$$

where $f(x,t+n\tau,\alpha)$ is given by (1.2.10). Finally, set

$$u^{n+1}(x) = If^n(x,\tau,\cdot), \quad x \in \bar{\mathfrak{D}}\ . \tag{1.2.14}$$

The problem (1.2.13) for f may be solved explicitly:

$$\begin{aligned} f^n(x,t,\alpha) = f^n\Big(&x-v(\alpha)[t-\max(0,\inf\{\xi \mid \alpha \in C(x,t-\xi)\})], \\ &\max(0,\inf\{\xi \mid \alpha \in C(x,t-\xi)\}),\alpha\Big)\ , \end{aligned} \tag{1.2.15}$$

$x \in \bar{\mathfrak{D}}$, $t \geq 0$. As in the one-dimensional case, the exact solution (1.2.15) may be approximated numerically by means of quadratures involving only non-negative quantities.

3. Burgers and Korteweg-de Vries Equations

The method described may be used to solve more general equations, and we indicate how one would proceed with some representative examples. The extension to still other types of equations will then be obvious.

Let us consider "Burgers' equation" (Reference 4)

$$u_t + \nabla \cdot (F(u)) = \nu \, \Delta \, u, \quad x \in \mathfrak{D}, \quad t > 0 \qquad (1.3.1a)$$

with

$$u(x,0) = u_0(x), \quad x \in \bar{\mathfrak{D}} \ , \qquad (1.3.1b)$$

and

$$u(x,t) = \tilde{u}(x,t), \quad x \in \partial\mathfrak{D}, \ \ t \in (0,T] \ . \qquad (1.3.1c)$$

(A more physically motivated generalization of Burgers' equation to a system of equations for a higher-dimensional flow, with the flow driven by the requirements of conservation of mass and momentum, will be treated in a later report on this work.)

As above, let $u_<$ satisfy

$$\begin{aligned} u_< &\le u_0(x), \quad x \in \bar{\mathfrak{D}} \ , \\ u_< &\le \tilde{u}(x,t), \quad x \in \partial\mathfrak{D}, \quad t \in (0,T] \ . \end{aligned} \qquad (1.3.2)$$

$u^0(x)$ is given by (1.2.12). Let $f^n(x,t,\alpha)$ satisfy the equation (v given by (1.2.2))

$$f_t^n + v(\alpha) \cdot \nabla f^n = \nu \, \Delta \, f^n, \quad x \in \mathfrak{D}, \quad t > 0 \qquad (1.3.3a)$$

with $f^n(x,0,\alpha)$ given by (1.2.13a) and the boundary condition

$$f^n(x,t,\alpha) = D(\tilde{u}(x,n\tau+t);\alpha), \quad x \in \partial\mathfrak{D}, \quad t > 0. \qquad (1.3.3b)$$

$D(u;\alpha)$ is given by (1.2.5). $u^{n+1}(x)$ is given in terms of $f^n(x,\tau,\alpha)$ by (1.2.14).

Note that we may combine (1.2.14) with the solution of equation (1.3.3a) subject to the initial condition (1.2.13a) and boundary condition (1.3.3b) approximately as follows:

$$u^{n+1}(x) \sim h(x,\tau) \tag{1.3.4}$$

where

$$\begin{aligned} h_t &= \nu \, \Delta \, h, \quad x \in \mathfrak{D}, \quad t > 0 \, , \\ h(x,0) &= Ig(x,\tau,\cdot), \quad x \in \bar{\mathfrak{D}} \, , \\ h(x,t) &= \tilde{u}(x,n\tau+t), \quad x \in \partial\mathfrak{D}, \quad t > 0 \, , \end{aligned} \tag{1.3.5}$$

and

$$\begin{aligned} & g_t + v(\alpha) \cdot \nabla g = 0, \quad x \in \mathfrak{D}, \quad t > 0, \\ & g(x,0,\alpha) = f^n(x,0,\alpha) \, , \\ & g(x,t,\alpha) = D(\tilde{u}(x,n\tau+t);\alpha), \quad x \in \partial\mathfrak{D}, \ \alpha \in C'(x,\xi) \quad \text{for all} \quad \xi > 0, \ t > 0 \, . \end{aligned} \tag{1.3.6}$$

The device of approximately solving the initial-boundary value problem for f^n and integrating to get u^{n+1} by solving (1.3.6) and (1.3.5) is analogous to the method of "fractional time steps". If $\mathfrak{D}$ is all space, this method becomes exact, because of the commutativity of the operators $\nu\Delta$ and $v(\alpha)\cdot\nabla$. (Solution of either initial value problem (1.3.6) or (1.3.5) is then equivalent to multiplying the spatial Fourier transform of the dependent function by an algebraic factor.) Thus, if we replace, in the algorithm (1.2.12), (1.3.3a), (1.2.13a), (1.3.3b), and (1.2.14), equations (1.3.3a), (1.3.3b), and (1.2.14) by (1.3.6), (1.3.5), and (1.3.4), so that the modified algorithm is described by (1.2.12), (1.2.13a), (1.3.6), (1.3.5), and (1.3.4), the solution of (1.3.1) is seen to be effected through a sequence of convections of non-interacting streams, interactions of these streams, and diffusions. This is similar to the description given at the end of section 1.1 of the algorithm (1.1.12)-(1.1.14), except that now a diffusive process is interspersed with the others.

Consider the "Korteweg-de Vries equation" (Reference 1)

$$u_t + \nabla \cdot (F(u)) + \beta \, \Delta \, u_x = 0, \quad \beta > 0, \quad x \in \mathfrak{D}, \quad t > 0 \, , \tag{1.3.7a}$$

with

$$u(x,0) = u_0(x), \quad x \in \bar{\mathfrak{D}} \, , \tag{1.3.7b}$$

and

$$u(x,t) = \tilde{u}(x,t), \quad x \in \partial\mathfrak{D}, \quad t \in (0,T] \quad , \tag{1.3.7c}$$

$$u_x(x,t) = \tilde{u}_x(x,t), \quad x \in \partial\mathfrak{D}^+, \quad t \in (0,T] \quad , \tag{1.3.7d}$$

where

$$\partial\mathfrak{D}^+ \equiv \{\xi \in \partial\mathfrak{D} \mid n(\xi) \cdot \vec{\iota} > 0\} \tag{1.3.8}$$

and $n(\xi)$ is the unit outward normal to $\partial\mathfrak{D}$ at ξ.

Proceeding as above, we will generate functions $u^n(x)$, $x \in \bar{\mathfrak{D}}$, $n \geq 0$, which are intended to approximate $u(x,n\tau)$. We start with $u^0(x)$ given by (1.2.12). With u^n calculated, we find a number $u^n_<$ such that

$$u^n_< \leq u^n(x), \quad x \in \bar{\mathfrak{D}} \quad ,$$

$$u^n_< \leq \tilde{u}(x,t), \quad x \in \partial\mathfrak{D}, \quad t \in (0,T] \quad . \tag{1.3.9}$$

We solve the equation

$$f^n_t + v(\alpha) \cdot \nabla f^n + \beta \, \Delta \, f^n_x = 0 \tag{1.3.10a}$$

with initial condition (1.2.13a), and boundary conditions (1.3.3b) and

$$f^n_x(x,t,\alpha) = \tilde{u}_x(x,t+n\tau) \, \delta(\alpha - \tilde{u}(x,n\tau+t)), \quad x \in \partial\mathfrak{D}^+, \quad t > 0 \quad . \tag{1.3.10b}$$

$v(\alpha)$ is given by (1.2.2). In the operators D and I given by (1.2.5) and (1.2.4), $u_<$ is replaced by $u^n_<$. Finally, u^{n+1} is calculated from (1.2.14).

Just as we could replace (1.3.3) and (1.2.14) in the algorithm for Burgers' equation by (1.3.4)-(1.3.6), as in a method of fractional time steps, we may replace (1.3.10) and (1.2.14) in this algorithm by two boundary value problems, one involving convection only, and the other dispersion only:

$$u^{n+1}(x) \sim h(x,\tau) \qquad (1.3.11)$$

where

$$h_t + \beta \Delta h_x = 0, \qquad x \in \mathfrak{D}, \qquad t > 0 \,, \qquad (1.3.12a)$$

$$h(x,0) = Ig(x,\tau,\cdot), \qquad x \in \bar{\mathfrak{D}} \,, \qquad (1.3.12b)$$

$$h(x,t) = \tilde{u}(x,n\tau+t), \qquad x \in \partial\mathfrak{D} \,, \qquad t > 0 \,, \qquad (1.3.12c)$$

$$h_x(x,t) = \tilde{u}_x(x,t+n\tau), \qquad x \in \partial\mathfrak{D}^+, \qquad t > 0, \qquad (1.3.12d)$$

and g satisfies (1.3.6). In this case the solution of (1.3.7) takes place through a sequence of convective, interactive, and dispersive processes.

The number $u^n_<$ is generally dependent on n in (1.3.9), since <u>a priori</u> bounds on u^n are not obtainable as easily as for the problems previously considered. In general solutions of (1.3.7) obey neither a maximum nor a minimum principle (References 9 and 12). This is related to the fact that, whereas each of the components (1.3.5) and (1.3.6) of the algorithm for Burgers' equation possesses a maximum and minimum principle, the same is not true for (1.3.12). In the one-dimensional case with $\mathfrak{D} = R^1$, we get maximum and minimum principles in one direction only, in the sense that if $h(x,0) \equiv 0$ for $x > x_0$, then $h(x,t)$ lies in the range of $h(x,0)$ for $x > x_0$ and $t > 0$.

4. Equations with Non-Constant Coefficients and Systems of Equations

Let us restrict ourselves to the case where the governing partial differential equation holds over all space, so that we can focus attention on solution of the initial value problem, and not on satisfaction of the boundary conditions. Taking $u_< = 0$ in (1.1.11) without any essential loss of generality, we note that the algorithm (1.1.12)-(1.1.14) is nothing more nor less than an approximate solution of (1.1.2), with initial condition (1.1.1b), recast in the following form:

$$u(x,t) = \lim_{\alpha \uparrow \infty} u(x,t,\alpha), \quad x \in R^1, \quad t > 0, \tag{1.4.1a}$$

$$u_0(x,t) \equiv 0, \quad x \in R^1, \quad t > 0\ , \tag{1.4.1b}$$

$$u(x,0,\alpha) = \min(u_0(x),\alpha), \quad x \in R^1, \quad 0 \le \alpha < \infty\ , \tag{1.4.1c}$$

$$\left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_t + \left(v(u(\alpha))\,\frac{\partial u(\alpha)}{\partial \alpha}\right)_x = 0, \quad x \in R^1, \quad t > 0, \quad 0 \le \alpha < \infty\ . \tag{1.4.1d}$$

In particular, (1.4.1d) is obtained by differentiating (1.1.2a) with respect to α, and the intermediate solutions $u(\alpha)$ are chosen in such a way that either the characteristics

$$\frac{dx}{dt} = v(u(\alpha)) \tag{1.4.2}$$

of (1.4.1d) are straight lines, because $u(\alpha)$ and $v(u(\alpha))$ are constant on them ($u(\alpha) = \alpha$), or else $\frac{\partial u(\alpha)}{\partial \alpha} = 0$ on them. Thus, we may view the algorithm (1.1.12)-(1.1.14) as a way of building up the solution $u(x,t)$, starting with the trivial solution $u_0(x,t)$, by means of a continuum of perturbations, dependent on the parameter α.

Similarly, the algorithms of sections 1.2 and 1.3 may be considered to be obtained by formal differentiation of the equations (1.2.1a), (1.3.1a), and (1.3.7a), respectively, with respect to α. The algorithm (1.2.12), (1.3.9), (1.2.13a), (1.3.6), (1.3.12), and (1.3.11) thus appears as an approximate solution to (1.3.7) by means of a continuum of perturbations on the solution $u(x,t) \equiv 0$, but it should be emphasized that this is not the same as a solution built up out of a formal expansion of the solution $u(\vec{x},t,\epsilon)$ of

$$u_t(\vec{x},t,\epsilon) + \epsilon\, \nabla \cdot F(u(\vec{x},t,\epsilon)) + \beta\, \Delta\, u_x(\vec{x},t,\epsilon) = 0 \tag{1.4.3}$$

in powers of ϵ, a procedure which has been shown to fail to yield solitary waves (Reference 10).

In this vein we might approach the problem of solving

$$G(u, u_t, \nabla u, x, t) = 0 \quad , \tag{1.4.4}$$

subject to given initial conditions, as a problem of determining a family of functions $u(x,t;\alpha)$, each satisfying (1.4.4), with initial conditions dependent on α. By differentiating (1.4.4) with respect to α, we get the linear equation

$$G_{u(\alpha)} \frac{\partial u(\alpha)}{\partial \alpha} + G_{u_t(\alpha)} \left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_t + G_{\nabla u(\alpha)} \cdot \nabla \frac{\partial u(\alpha)}{\partial \alpha} = 0 \; . \tag{1.4.5}$$

The characteristics of this equation are

$$\frac{dt}{ds} = G_{u_t(\alpha)}, \quad \frac{dx}{ds} = G_{\nabla u(\alpha)} \quad , \tag{1.4.6a}$$

and on them $\frac{\partial u(\alpha)}{\partial \alpha}$ satisfies

$$\frac{d}{ds} \frac{\partial u(\alpha)}{\partial \alpha} + G_{u(\alpha)} \frac{\partial u(\alpha)}{\partial \alpha} = 0 \; . \tag{1.4.6b}$$

If $x \in R^n$, we have $n + 2$ equations (1.4.6), and the coefficients of these equations generally depend on the $2n + 3$ quantities x, t, $u(\alpha)$, $u_t(\alpha)$, and $\nabla u(\alpha)$. If we desire these equations to have constant coefficients, for each α, we can satisfy that requirement if we pick α as a measure on the Borel sets in R^m where $m = n + 2$, and think of $u(\alpha)$ as a corresponding signed measure, where $\frac{\partial u(\alpha)}{\partial \alpha}$ is now to be thought of as a Radon-Nikodym derivative (Reference 7). As a practical means of solving (1.4.4), this procedure does not seem to be generally promising; we indicate some more useful approaches below.

Higher-order equations than (1.4.4) can be solved. As an example, one might solve

$$u_t = uu_{xx}, \quad u(x,0) > 0 \,, \tag{1.4.7a}$$

by solving at each time

$$\left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_t = u(\alpha)\left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_{xx} + \frac{\partial u(\alpha)}{\partial \alpha}\,(u(\alpha))_{xx} \tag{1.4.7b}$$

and choosing $u(\alpha)$ so that either $\frac{\partial u(\alpha)}{\partial \alpha} = 0$ or $u(\alpha) = \alpha$ = constant.

For the equation

$$u_t + xu_x = 0 \quad , \tag{1.4.8a}$$

we get

$$\left(\frac{\partial u}{\partial \alpha}\right)_t + x\left(\frac{\partial u}{\partial \alpha}\right)_x = 0 \quad , \tag{1.4.8b}$$

and we want to choose $u(\alpha)$ so that either $\frac{\partial u}{\partial \alpha} = 0$ or $x = \alpha$ = constant. Note that this equation is not in conservation form. A conservative equation is obtained by writing $u' = u_x$ and

$$u'_t + (xu')_x = 0 \quad , \tag{1.4.9a}$$

$$\left(\frac{\partial u'}{\partial \alpha}\right)_t + x\left(\frac{\partial u'}{\partial \alpha}\right)_x + \frac{\partial u'}{\partial \alpha} = 0 \quad . \tag{1.4.9b}$$

For the equation

$$u_t + xuu_x = 0 \quad , \tag{1.4.10a}$$

we obtain

$$\left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_t + xu(\alpha)\left(\frac{\partial u(\alpha)}{\partial \alpha}\right)_x + xu_x(\alpha)\,\frac{\partial u(\alpha)}{\partial \alpha} = 0 \quad . \qquad (1.4.10b)$$

To solve this, we take α as a two-dimensional measure, and choose $u(\alpha)$ so that $\frac{\partial u}{\partial \alpha} = 0$ or x = constant and $u(\alpha)$ = constant.

It may be that an equation

$$u_t = \nabla \cdot G_1 + \nabla \cdot G_2 \qquad (1.4.11)$$

can be solved more efficiently by means of a method of "fractional time steps", by solving at each time the equations

$$u_t = \nabla \cdot G_1 \qquad (1.4.12a)$$

and

$$u_t = \nabla \cdot G_2 \qquad (1.4.12b)$$

successively. The individual equations (1.4.12a,b) may be solved more easily by the methods described here, than can the original equation (1.4.11). For example, the solution of

$$u_t + \left((x+\frac{u}{2})u\right)_x = 0 \qquad (1.4.13a)$$

may be reduced to the successive solution of

$$u_t + (xu)_x = 0 \qquad (1.4.13b)$$

and

$$u_t + uu_x = 0 \ . \qquad (1.4.13c)$$

Since quite general nonlinear systems of equations may be reduced to first order quasilinear systems which are homogeneous in the derivatives and do not involve the independent variables explicitly (Reference 6), we may think of the equations above in which the independent variables appear explicitly as special cases of homogeneous quasilinear first order systems of equations. As as example of a system of the latter genre, consider the equations

$$G_i(u^{(1)},u^{(2)},u_t^{(1)},u_t^{(2)},\nabla u^{(1)},\nabla u^{(2)}) = 0, \qquad i=1,2 , \tag{1.4.14}$$

where the G_i depend linearly on $u_t^{(1)}$, $u_t^{(2)}$, $\nabla u^{(1)}$, and $\nabla u^{(2)}$. If the system (1.4.14) is hyperbolic, we may pick α as a two-dimensional measure such that we either have $\frac{\partial u^{(1)}}{\partial \alpha} = 0$ and $\frac{\partial u^{(2)}}{\partial \alpha} = 0$ or $u^{(1)}(\alpha)$ = constant and $u^{(2)}(\alpha)$ = constant.

It should be emphasized that such a formal scheme to obtain solutions says nothing about the well-posedness of the original problem, unless we know something about the well-posedness of all the linear problems which are solved for $\frac{\partial u^{(1)}}{\partial \alpha}$ and $\frac{\partial u^{(2)}}{\partial \alpha}$. For example, the system

$$u_t^{(1)} + \frac{\partial}{\partial x}\left(u^{(1)}\ e^{u^{(2)}}\right) = 0 ,$$

$$u_t^{(2)} - \frac{\partial}{\partial x}\left(u^{(2)}\ e^{u^{(1)}}\right) = 0 , \tag{1.4.15}$$

is only hyperbolic if

$$\left(e^{u^{(1)}} + e^{u^{(2)}}\right)^2 - 4\ u^{(1)}\ u^{(2)}\ e^{u^{(1)}+u^{(2)}} > 0. \tag{1.4.16}$$

CHAPTER TWO

CONVERGENCE OF THE ALGORITHM FOR A SPECIAL CASE

The principal result of this chapter will be a proof of convergence of the algorithm (1.1.12)-(1.1.14) in the case when $x_0 \to -\infty$, $x_1 \to +\infty$. To carry out the proof, we will have to suitably restrict the initial function $u_0(x)$ in (1.1.1b). In the first section of this chapter we discuss suitable function spaces for the study of convergence and introduce some operators which expedite the analysis of the algebraic structure of the algorithm (1.1.12)-(1.1.14). We will be able to reduce the study of convergence to the case when the initial function $u_0(x)$ is of a special type. In the second section we present an error estimate, and in order to get a particularly sharp result we will require that the "velocity" v(u) in (1.1.1a) be a monotonic function of u. Other cases may be treated, and we will show what sorts of error estimates may be expected for the algorithm (1.1.12)-(1.1.14) in those cases.

1. Function Spaces and Monotonic Operators

It is clear that we will need some property of continuity for the function $u_0(x)$, and cannot require, for example, that $u_0(x)$ just be bounded measurable. For instance, if we had

$$u_0(x) = \begin{cases} 2 & 2n\,\tau_0 \le x < (2n+1)\tau_0 \\ 0 & (2n+1)\tau_0 \le x < 2(n+1)\tau_0 \end{cases}, \quad n=0,\pm1,\pm2,\cdots, \tag{2.1.1}$$

the algorithm (1.1.12)-(1.1.14) will yield markedly different results according to whether we pick $\tau = \tau_0$ or $\tau = \tau_0/2$.

We shall require $u_0(x)$ to be almost uniformly continuous according to the following definition.

Definition: A function $g\colon R^1 \to R^1$ is called almost uniformly continuous (a.u.c.) if for every a and b such that $-\infty < a < b < \infty$ and $\epsilon > 0$, $\exists\, \delta = \delta(\epsilon,a,b)$ with

$$\int_a^b \left| g(x+\eta_1(x)\delta) - g(x+\eta_2(x)\delta)\right| dx < \epsilon \tag{2.1.2a}$$

for all $\eta_1(x)$, $\eta_2(x)$ such that

$$|\eta_1(x)| \le 1, \ |\eta_2(x)| \le 1 \quad . \tag{2.1.2b}$$

A more general class of $u_0(x)$ will be mentioned later.

At this point, let us recall the operators D and I introduced in (1.1.17):

$$D(u(\cdot);\alpha) = \begin{cases} 0 & \alpha < u_< \\ 1 & u_< \le \alpha \le u(\cdot) \\ 0 & \alpha > u(\cdot) \end{cases} \tag{2.1.3a}$$

and

$$IF(\cdot,\cdot) = u_< + \int_{-\infty}^{\infty} f(\cdot,\alpha)d\alpha \quad , \tag{2.1.3b}$$

where

$$u_< \le u_0(x), \qquad x \in R^1 \quad . \tag{2.1.3c}$$

Also, recall the definition of V in (1.1.26) and (1.1.28).

Lemma 2.1.1: Let $u^n(x)$ be given by (1.1.12)-(1.1.14). If

$$\int_a^b |u_0(x+\eta_1(x)\delta) - u_0(x+\eta_2(x)\delta)|dx < \epsilon \tag{2.1.4}$$

for all η_1 and η_2 satisfying (2.1.2b), then

$$\int_{a+n\tau V}^{b-n\tau V} |u^n(x+\eta_1(x)\delta) - u^n(x+\eta_2(x)\delta)|dx < \epsilon \tag{2.1.5}$$

for all such η_1, η_2. In other words, u_0 a.u.c. $\Rightarrow u^n$ a.u.c.

Proof: It is obviously sufficient to prove (2.1.5) for the case n = 1. From (1.1.14) and (1.1.15),

$$|u^1(x+\eta_1(x)\delta)-u^1(x+\eta_2(x)\delta)| \le \int_{-\infty}^{\infty} |f^0(x+\eta_1(x)\delta,\tau,\alpha)-f^0(x+\eta_2(x)\delta,\tau,\alpha)|\,d\alpha$$

$$= \int_{-\infty}^{\infty} |f^0(x+\eta_1(x)\delta-\tau v(\alpha),0,\alpha)-f^0(x+\eta_2(x)\delta-\tau v(\alpha),0,\alpha)|\,d\alpha \quad . \quad (2.1.6)$$

Then, by (1.1.13b) and (1.1.12),

$$\int_{a+V\tau}^{b-V\tau} |u^1(x+\eta_1(x)\delta)-u^1(x+\eta_2(x)\delta)|\,dx$$

$$\le \int_{-\infty}^{\infty}\int_{a+V\tau}^{b-V\tau} |f^0(x+\eta_1(x)\delta-\tau v(\alpha),0,\alpha)-f^0(x+\eta_2(x)\delta-\tau v(\alpha),0,\alpha)|\,dx\,d\alpha$$

$$= \int_{-\infty}^{\infty}\int_{a+V\tau-\tau v(\alpha)}^{b-V\tau-\tau v(\alpha)} |f^0(x+\eta_1(x+\tau v(\alpha)),0,\alpha)-f^0(x+\eta_2(x+\tau v(\alpha)),0,\alpha)|\,dx\,d\alpha$$

$$\le \int_a^b \int_{-\infty}^{\infty} |f^0(x+\eta_1(x+\tau v(\alpha)),0,\alpha)-f^0(x+\eta_2(x+\tau v(\alpha)),0,\alpha)|\,d\alpha\,dx$$

$$\le \int_a^b \left(\sup_{y\in[x-\delta,x+\delta]} u_0(y) - \inf_{y\in[x-\delta,x+\delta]} u_0(y) \right) dx$$

$$\le \int_a^b |u_0(x+\eta_3(x)\delta)-u_0(x+\eta_4(x)\delta)|\,dx + 2\,\theta(b-a) \;, \qquad (2.1.7)$$

where we pick $\eta_3(x)$ and $\eta_4(x)$ to satisfy the same requirement (2.1.2b) as η_1 and η_2, and

$$u_0(x+\eta_3(x)\delta) \ge \sup_{y\in[x-\delta,x+\delta]} u_0(y) - \theta \;,$$

$$u_0(x+\eta_4(x)\delta) \le \inf_{y\in[x-\delta,x+\delta]} u_0(y) + \theta \;, \qquad (2.1.8)$$

with $\theta > 0$. Letting $\theta \downarrow 0$ and using (2.1.4), we obtain (2.1.5) for $n = 1$.

Before proceeding further, let us introduce operators $\Sigma^{\pm}_{\eta}$, where η is a function of α:

$$(\Sigma^{+}_{\eta} f)(x,\cdot) = \sup_{y \in I(x,\eta)} f(y,\cdot) \quad , \tag{2.1.9a}$$

$$(\Sigma^{-}_{\eta} f)(x,\cdot) = \inf_{y \in I(x,\eta)} f(y,\cdot) \quad , \tag{2.1.9b}$$

where

$$I(x,\eta) = [\min(x,x+\eta(\cdot)),\ \max(x,x+\eta(\cdot))] \quad . \tag{2.1.9c}$$

The following properties can readily be verified for the operators $\Sigma^{\pm}_{\eta}$:

$$\Sigma^{+}_{\eta_1} \Sigma^{+}_{\eta_2} = \Sigma^{+}_{\eta_1^+ + \eta_2^+} \Sigma^{+}_{\eta_1^- + \eta_2^-} \quad , \tag{2.1.10a}$$

$$\Sigma^{-}_{\eta_1} \Sigma^{-}_{\eta_2} = \Sigma^{-}_{\eta_1^+ + \eta_2^+} \Sigma^{-}_{\eta_1^- + \eta_2^-} \quad , \tag{2.1.10b}$$

where

$$\eta^{+}(\cdot) \equiv \max(\eta(\cdot),0) \quad ,$$
$$\tag{2.1.11}$$
$$\eta^{-}(\cdot) \equiv \min(\eta(\cdot),0) \quad .$$

In particular, it follows from (2.1.10) that

$$\Sigma^{+}_{\eta_1} \Sigma^{+}_{\eta_2} = \Sigma^{+}_{\eta_2} \Sigma^{+}_{\eta_1} \quad , \tag{2.1.12a}$$

$$\Sigma^-_{\eta_1} \Sigma^-_{\eta_2} = \Sigma^-_{\eta_2} \Sigma^-_{\eta_1} , \qquad (2.1.12b)$$

and if $\eta_1 \eta_2 \geq 0$,

$$\Sigma^+_{\eta_1} \Sigma^+_{\eta_2} = \Sigma^+_{\eta_1 + \eta_2} , \qquad (2.1.13a)$$

$$\Sigma^-_{\eta_1} \Sigma^-_{\eta_2} = \Sigma^-_{\eta_1 + \eta_2} . \qquad (2.1.13b)$$

Also

$$\Sigma^+_{\eta_1} \Sigma^-_{\eta_2} \leq \Sigma^-_{\eta_2} \Sigma^+_{\eta_1} , \qquad (2.1.14)$$

and

$$\Sigma^-_{-\eta} \Sigma^+_{\eta} \geq 1 , \qquad (2.1.15b)$$

$$\Sigma^+_{\eta} \Sigma^-_{-\eta} \leq 1 . \qquad (2.1.15b)$$

If $|\eta| \leq |\delta|$, we get

$$\Sigma^-_{-\eta} \Sigma^+_{\eta} \Sigma^-_{\delta} = \Sigma^-_{\delta} \qquad (2.1.16a)$$

and

$$\Sigma^+_{\eta} \Sigma^-_{-\eta} \Sigma^+_{\delta} = \Sigma^+_{\delta} . \qquad (2.1.16b)$$

In terms of the Σ's we can define

$$S^{\pm}_{\eta} = \Sigma^{\pm}_{\eta} \Sigma^{\pm}_{-\eta} . \qquad (2.1.17)$$

Given a function g(x) and a number $\delta > 0$ independent of α, we can also define

$$(S_\delta^\pm g)(x) = (IS_\delta^\pm Dg)(x) \quad . \tag{2.1.18}$$

In other words

$$(S_\delta^+ g)(x) = \sup_{y\in[x-\delta,x+\delta]} g(y) \quad , \tag{2.1.19a}$$

$$(S_\delta^- g)(x) = \inf_{y\in[x-\delta,x+\delta]} g(y) \quad . \tag{2.1.19b}$$

Finally, let us introduce the shift operator T_η:

$$(T_\eta f)(x,\cdot) = f(x-\eta(\cdot),\cdot) \quad . \tag{2.1.20}$$

The following relations among these operators can easily be established:

$$\Sigma_{-\eta}^+ \Sigma_{-\eta}^- \le T_\eta \le \Sigma_{-\eta}^- \Sigma_{-\eta}^+ \quad , \tag{2.1.21a}$$

$$T_\eta S_{\eta/2}^- = \Sigma_{-\eta}^- \Sigma_{-\eta}^+ S_{\eta/2}^- \quad , \tag{2.1.21b}$$

and

$$T_\eta S_{\eta/2}^+ = \Sigma_{-\eta}^+ \Sigma_{-\eta}^- S_{\eta/2}^+ \quad . \tag{2.1.21c}$$

We get commutativity between the Σ's and T:

$$T_{\eta_1} \Sigma_{\eta_2}^\pm = \Sigma_{\eta_2}^\pm T_{\eta_1} \tag{2.1.22}$$

and we note

$$\Sigma^{\pm}_{-\eta} = T_{\eta}\, \Sigma^{\pm}_{\eta} \quad , \tag{2.1.23}$$

so that

$$S^{\pm}_{\eta} = T_{\eta}\, \Sigma^{\pm}_{2\eta} = T_{-\eta}\, \Sigma^{\pm}_{-2\eta} \quad . \tag{2.1.24}$$

All the operators introduced are monotonic. That is,

$$g_1 \geq g_2 \Rightarrow 0g_1 \geq 0g_2, \quad 0 = D, \quad S^{+}_{\delta}, \quad \text{or} \quad S^{-}_{\delta} \tag{2.1.25a}$$

and

$$f_1 \geq f_2 \Rightarrow 0f_1 \geq 0f_2, \; 0 = I, \; T_{\eta}, \; S^{+}_{\eta}, \; S^{-}_{\eta}, \; \Sigma^{+}_{\eta}, \; \text{or} \; \Sigma^{-}_{\eta} \quad . \tag{2.1.25b}$$

From the definitions it follows that

$$IS^{+}_{\delta} > S^{+}_{\delta}\, I \quad , \tag{2.1.26a}$$

$$DS^{+}_{\delta} = S^{+}_{\delta}\, D \quad , \tag{2.1.26b}$$

$$T_{\eta}\, S^{+}_{\delta} = S^{+}_{\delta}\, T_{\eta} \tag{2.1.26c}$$

and

$$IS^{-}_{\delta} \leq S^{-}_{\delta}\, I \quad , \tag{2.1.27a}$$

$$DS^{-}_{\delta} = S^{-}_{\delta}\, D \quad , \tag{2.1.27b}$$

$$T_{\eta}\, S^{-}_{\delta} = S^{-}_{\delta}\, T_{\eta} \quad . \tag{2.1.27c}$$

Also,

$$S_\eta^- \le T_\eta \le S_\eta^+ \tag{2.1.28a}$$

and

$$S_\delta^- D \le T_\eta D \le S_\delta^+ D \tag{2.1.28b}$$

where $|\eta(\alpha)| \le \delta$ for all α.

In terms of these operators, with the help of (1.1.15), we can state the algorithm (1.1.12)-(1.1.14) as

$$u^{n+1} = IT_{v\tau} Du^n \quad . \tag{2.1.29}$$

Thus, the algorithm proceeds entirely through the application of monotonic operators. If

$$u_0(x) \ge \tilde{u}_0(x) \quad , \tag{2.1.30a}$$

and we denote the respective approximate solutions by $u^n(x)$ and $\tilde{u}^n(x)$ (with the same $u_<$ for $u_0(x)$ and $\tilde{u}_0(x)$) we get

$$u^n(x) \ge \tilde{u}^n(x) \quad . \tag{2.1.30b}$$

This sort of monotonicity result also holds for the problem (1.1.5) posed on a finite domain, provided we have the appropriate monotonicity required of the corresponding boundary data. The result also holds for the algorithm (1.2.12)-(1.2.14) used to solve the problem (1.2.1), and for the algorithm used in the solution of (1.3.1). Assuming convergence of the algorithms for these problems, we conclude that the monotonicity carries over to the exact solutions of (1.1.5), (1.2.1), and (1.3.1) generated by the algorithms. At this juncture we should reemphasize the point made in section 1.1, that the solution to (1.1.1) computed by our procedure is not necessarily the desired one in terms of any underlying physical rationale. For example, the one-dimensional flow of an inviscid pressureless fluid is governed by an equation of type (1.1.1) with u the velocity and $v(u) = u$. However, the underlying physical laws are conservation of mass and momentum, and u is

really coupled to a density field ρ. For general initial density fields, we do not have monotonic dependence of $u(x,t)$ on $u(x,0)$. Hence any algorithm with that property will have to be in error, in general. In a later part of this work we will present a physically correct algorithm to solve

$$u_t + uu_x = 0$$

when u is the velocity of a one-dimensional flow.

Returning to the operators $S_\delta^\pm$, we see that

$$\begin{aligned} S_\delta^+ S_\delta^- &\le 1 \ , \\ S_\delta^- S_\delta^+ &\ge 1 \ , \\ S_\delta^+ S_\delta^- S_\delta^+ &= S_\delta^+ \ , \\ S_\delta^- S_\delta^+ S_\delta^- &= S_\delta^- \ , \end{aligned} \tag{2.1.31}$$

The operator $S_\delta^+ S_\delta^-$ "flattens out" all peaks of a function $g(x)$ by removing portions of width $< 2\delta$. Similarly, $S_\delta^- S_\delta^+$ "flattens out" the troughs by filling in the portions of width $< 2\delta$.

Upon application of equations (2.1.26), (2.1.27), and (2.1.31), we see that the following inequalities can be established by induction in n:

$$\begin{aligned} S_\delta^-(IT_{v\tau}D)^n S_\delta^+ u_0 &\ge (S_\delta^- S_{2\delta}^+ S_\delta^- IT_{v\tau} D)^n S_\delta^- S_{2\delta}^+ S_\delta^- u_0 \\ &\ge S_\delta^+ (IT_{v\tau}D)^n S_\delta^- u_0 \ . \end{aligned} \tag{2.1.32}$$

The significance of (2.1.32) is that it gives bounds on the quantities w^n computed by the algorithm

$$w^0 = S_\delta^- S_{2\delta}^+ S_\delta^- u_0 \ , \tag{2.1.33a}$$

$$w^{n+1} = S_\delta^- S_{2\delta}^+ S_\delta^- IT_{V\tau} D w^n \quad . \tag{2.1.33b}$$

When u_0 is a.u.c., we can use these bounds to show that w^n is close to u^n, as we shall do below. Assuming that this is the case, we see then that we may equally well use the algorithm (2.1.33), as use the original algorithm (1.1.12)-(1.1.14) (or (2.1.29)) to solve the problem. Furthermore, each of the quantities w^n, $n \geq 0$, has the property that all peaks and troughs are of width $\geq 2\delta$. This fact will permit us, in the convergence proof given in the next section, to restrict our attention to portions of the profile $w^o(x)$ which are monotonically non-increasing or non-decreasing. That is, if for example (ξ_0, ξ_1) is the interval between a flattened peak and trough of u_0, then for $0 \leq n\tau \leq 2\delta/V$ the function $w^n(x)$, $x \in [\xi_0, \xi_1]$, will be either non-increasing or non-decreasing, according as w^o is. It is clear that if we can establish convergence and error estimates as $\tau \downarrow 0$ for such intervals $(x,t) \in [\xi_0, \xi_1] \times [0, 2\delta/V]$, then we can do so for all $(x,t) \in (-\infty,\infty) \times [0,\infty)$.

To show that w^n is close to u^n if δ is sufficiently small, we use (2.1.32) in the form

$$\theta^n \geq w^n \geq \psi^n \quad , \tag{2.1.34}$$

where

$$\theta^o = S_\delta^+ u_0 \quad , \tag{2.1.35a}$$

$$\theta^{n+1} = IT_{V\tau} D \theta^n \quad , \tag{2.1.35b}$$

and

$$\psi^o = S_\delta^- u_0 \tag{2.1.36a}$$

$$\psi^{n+1} = IT_{V\tau} D \psi^n \quad . \tag{2.1.36b}$$

Because of the monotonicity result (2.1.30), we see from (2.1.35) and (2.1.36) that

$$\theta^n \geq u^n \geq \psi^n \tag{2.1.37}$$

and hence

$$|w^n - u^n| \le \theta^n - \psi^n \quad . \tag{2.1.38}$$

Because of the monotonic dependence of θ^n and ψ^n on the initial data θ^o and ψ^o, and the fact, which follows from the definition of V, that all non-zero contributions to θ^n and ψ^n in $[a+n\tau V, b-n\tau V]$ have originated as contributions to θ^o and ψ^o somewhere in $[a,b]$, we get

$$\int_{a+n\tau V}^{b-n\tau V} (\theta^n - \psi^n)dx \le \int_a^b (\theta^o - \psi^o)dx \quad . \tag{2.1.39}$$

From (2.1.38), (2.1.39), and the initial conditions (2.1.35a) and (2.1.36a), we get the following result.

Lemma 2.1.2: If $u_o(x)$ is a.u.c. and satisfies (2.1.2) and if w^n is given by the algorithm (2.1.33) and u^n is given by (1.1.12)-(1.1.14), then

$$\int_{a+n\tau V}^{b-n\tau V} |w^n(x) - u^n(x)|dx \le \epsilon(\delta) \quad , \tag{2.1.40}$$

where we denote the right hand side of (2.1.2a) by $\epsilon(\delta,a,b)$.

With the aid of the operators introduced, we can also find the effect on the approximate solution $u^n(x)$ of (1.1.1) due to a perturbation of the "velocity" $v(u)$, when $u_o(x)$ is a.u.c. On the assumption that the approximate solution converges to an exact solution, the same error estimate will carry over to the exact solution (with $n\tau$ replaced by t and $u^n(x)$ replaced by $u(x,t)$).

Lemma 2.1.3: In the case $x_o \to -\infty$, $x_1 \to +\infty$, let $u^n(x)$ be generated from $u_o(x)$ by equations (1.1.12)-(1.1.14), and let $\tilde{u}^n$ be generated from $u_o(x)$ by the same equations with $v(\alpha)$ replaced by $\tilde{v}(\alpha)$. Suppose $u_o(x)$ is a.u.c. and satisfies (2.1.2). Use the notation $\epsilon(\delta)$ in (2.1.40). Suppose $|v(\alpha)| \le V$ and $|\tilde{v}(\alpha)| \le V$ for all $\alpha \in [u_<, U]$, where U is given in (1.1.26). Then, with $|v(\alpha) - \tilde{v}(\alpha)| \le V'$ for all $\alpha \in [u_<, U]$, we have

$$\int_{a+n\tau V}^{b-n\tau V} |\tilde{u}^n(x)-u^n(x)|\,dx \le \epsilon(n\tau V') \quad . \qquad (2.1.41)$$

Proof: Using (2.1.29), we can write

$$u^n = (IT_{V\tau}D)^n\, u_0 \quad , \qquad (2.1.42a)$$

$$\tilde{u}^n = (IT_{\tilde{V}\tau}D)^n\, u_0 \quad . \qquad (2.1.42b)$$

Now, we can establish inductively that

$$\tilde{u}^n \le (IT_{V\tau}D)^n\, S^+_{n\tau V'}\, u_0 \quad . \qquad (2.1.43)$$

The case $n = 0$ *is obvious.* To get from n to $n + 1$, we note that

$$\tilde{u}^{n+1} = IT_{\tilde{V}\tau}\, D\, \tilde{u}^n \le IS^+_{V'\tau}\, T_{V\tau}\, D\, \tilde{u}^n \le IT_{V\tau}\, D\, S^+_{V'\tau}\, (IT_{V\tau}D)^n\, S^+_{n\tau V'}\, u_0$$

$$\le (IT_{V\tau}D)^{n+1}\, S^+_{(n+1)\tau V'}\, u_0 \quad ,$$

upon repeated application of (2.1.26). Similarly, we can show that

$$\tilde{u}^n \ge (IT_{V\tau}D)^n\, S^-_{n\tau V'}\, u_0 \quad . \qquad (2.1.44)$$

Let

$$\theta^0_n = S^+_{n\tau V'}\, u_0, \quad \psi^0_n = S^-_{n\tau V'}\, u_0 \quad , \qquad (2.1.45a)$$

$$\theta^n_n = (IT_{V\tau}D)^n\, \theta^0_n, \quad \psi^n_n = (IT_{V\tau}D)^n\, \psi^0_n \quad . \qquad (2.1.45b)$$

That is, θ^n_n and ψ^n_n are the functions generated at the nth time step for the unperturbed velocity $v(\alpha)$, from the initial functions

θ_n^o and ψ_n^o, respectively. By reasoning similar to that which led to (2.1.39),

$$\int_{a+n\tau V}^{b-n\tau V} (\theta_n^n - \psi_n^n)\,dx \le \int_a^b (\theta_n^o - \psi_n^o)\,dx \quad . \tag{2.1.46}$$

The monotonic dependence on the initial data also implies

$$\psi_n^n \le u^n \le \theta_n^n \quad . \tag{2.1.47}$$

From (2.1.47), (2.1.43), (2.1.44), (2.1.45), and (2.1.46), we get

$$\int_{a+n\tau V}^{b-n\tau V} |\tilde{u}^n(x) - u^n(x)|\,dx \le \int_a^b (S^+_{n\tau V},\ u_o - S^-_{n\tau V},\ u_o)\,dx \quad , \tag{2.1.48}$$

and this leads directly to (2.1.41), upon use of the almost uniform continuity of u_o and (2.1.2).

Thus if $v(\alpha)$ is piecewise continuous, we can replace it by a piecewise constant function $\tilde{v}$ with $|v-\tilde{v}|$ as small as we desire, and for a.u.c. initial data get as small a change in the solution $u^n(x)$ as we desire (in the L_1 sense).

By an argument similar to that used to establish lemma 2.1.3, we can prove the continuous dependence, in the L_1 sense, of the functions generated by (1.1.12)-(1.1.14) on the "time". We again use monotonicity and note that

$$S^-_{m\tau V}\ u^o \le u^m \le S^+_{m\tau V}\ u^o \quad . \tag{2.1.49}$$

There follows

$$\int_{a+n\tau V}^{b-n\tau V} |u^n(x) - u^{n+m}(x)|\,dx \le \epsilon(m\tau V) \quad . \tag{2.1.50}$$

A somewhat broader notion of continuity is that of "almost continuity" (a.c.)* of a function g, by which we mean that, for any $\epsilon > 0$, we can find a set S_ϵ of Lebesgue measure $< \epsilon$ such that g is uniformly continuous on $[a,b] - S_\epsilon$. We will find this type of continuity, suitably generalized to flows with other measures, to be especially important in the treatment of actual physical flows. Accordingly, there will be a more complete discussion of almost continuity in a later report in this work. Essentially, instead of removing a set of Lebesgue measure $< \epsilon$ from the domain, we will want to remove sets of small mass or action, depending on the precise type of continuity considered, from various domains of independent variables. Further on, when we treat the flows in a stochastic framework, still further enlargement of the notion of almost continuity will be in order. For the present, let us merely note that if g is a.c., then g is measurable, but the reverse implication does not hold. In addition, it follows from the definitions that if g is a.u.c., then it is a.c. The converse is not necessarily true, unless the set S_ϵ is dense on a set in $[a,b]$ whose measure $\downarrow 0$ as $\epsilon \downarrow 0$. For example, the converse is not true if g is 1 at all rational numbers in $[a,b]$ and 0 elsewhere. Then g is a.c. but not a.u.c. With a little reflection, we see that an analogue of lemma 2.1.1 can be proven when u_o is a.c.: u_oa.c. $\Rightarrow$ u^na.c. Similarly, the continuous dependence of the functions $u^n(x)$ generated by the algorithm (1.1.12)-(1.1.14) on the "velocity" $v(\alpha)$, stated by lemma 2.1.3 for a.u.c. initial data $u_o(x)$, can be proven for a.c. initial data, and the suitability of replacing the algorithm (1.1.12)-(1.1.14) by the algorithm (2.1.33), stated in lemma 2.1.2 for a.u.c. initial data, holds for a.c. initial data.

The types of continuity we have been considering in this section are weaker than that required in section 1.1 to show that the right hand sides of (1.1.33) and (1.1.37), for $0 < t \le \tau$, are $o(1)$ and $o(\tau)$, respectively, as $\tau \downarrow 0$. It would be tempting to conclude that $u_o(x)$ a.u.c. or a.c. implies that $u(x_o,t)$ is a.u.c. or a.c. in time for each x_o. This is not the case, however, as we can see from the following example. In the limit $\beta\gamma \downarrow 0$, and with $v(\alpha) = \alpha$, choose the initial profile near $x = 0$ to satisfy

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*Terminology suggested by Avron Douglis.

$$u_0(x) = \begin{cases} U - \beta x & 0 > x > -U\gamma \\ -U & 0 < x < U\gamma \\ U(1+\beta\gamma) & -U\gamma > x > -2U\gamma \\ -U - \beta(x-U\gamma) & U\gamma < x < 2U\gamma \end{cases} \quad . \tag{2.1.51}$$

Initially there is a jump 2U at x = 0, and it is easy to see that, if s(t) is the location at time t of the discontinuity in u(x,t), $\frac{ds}{dt}(0) = 0$. After a time 2γ, we get $s(2\gamma) = \frac{\beta\gamma^2}{2}U$, $u(s(2\gamma)^-,2\gamma) = U(1+\beta\gamma)$, $u(s(2\gamma)^+,2\gamma) = -U(1+\beta\gamma)$, and $\frac{ds}{dt}(2\gamma) = 0$. We can now adjust $u_0(x)$ so that s(t) moves back when $2\gamma < t < 4\gamma$, forward again when $4\gamma < t < 6\gamma$, and so on. After a time t, in the limit $\frac{\gamma}{t} \downarrow 0$, we get a jump $2Ue^{\frac{\beta t}{2}}$ at a position s(t) which oscillates between

$$\frac{\beta\gamma^2}{4}\left(1-e^{\frac{\beta t}{2}}\right)U \quad \text{and} \quad \frac{\beta\gamma^2}{4}\left(1+e^{\frac{\beta t}{2}}\right)U \quad .$$

Points in the interval $[0, \frac{\beta\gamma^2}{2} U]$ are included in all the oscillations. In particular, we cannot make any statement about the a.c. or a.u.c. of u(0,t) which will be valid independent of the choice of γ, although the initial profile $u_0(x)$ will be a.u.c., according to the definition (2.1.2), with $\epsilon(\delta)$ independent of γ. The values of x for which we get uniformly large variations in u(x,t) when t changes by an amount γ independent of x will, of course, have Lebesgue measure which goes to 0 as $\gamma \downarrow 0$, by the result (2.1.50).

(The fact that we cannot get <u>a priori</u> that $u(x_0,t)$ is a.u.c. or a.c. for each x_0 is related to our ability to <u>control</u> the free boundary problem. In particular, for the case $v(\alpha) = \alpha$, if we are given quantities J(t), s(t) for $0 \le t \le T$ with the following properties:

$$J(0) \ge 0 \quad , \tag{2.1.52a}$$

$$\frac{dJ}{dt}(t) \text{ bounded, non-negative, } 0 \le t \le T \ , \tag{2.1.52b}$$

$$\left|\frac{d^2 s}{dt^2}(t)\right| \le \frac{1}{2}\frac{dJ}{dt}, \quad 0 \le t \le T, \tag{2.1.52c}$$

then there is an initial profile $u_0(x)$, which is unique and differentiable in the intervals

$$s(T) - T\left[\frac{1}{2}J(T) + \frac{ds}{dt}(T)\right] \le x < 0 \tag{2.1.53a}$$

and

$$0 < x \le s(T) + T\left[\frac{1}{2}J(T) - \frac{ds}{dt}(T)\right], \tag{2.1.53b}$$

and which satisfies

$$x + u_0(x)T < s(T) \quad \text{for} \quad x < s(T) - T\left[\frac{1}{2}J(T) + \frac{ds}{dt}(T)\right], \tag{2.1.54a}$$

$$x + u_0(x)T > s(T) \quad \text{for} \quad x > s(T) + T\left[\frac{1}{2}J(T) - \frac{ds}{dt}(T)\right], \tag{2.1.54b}$$

such that the solution of (1.1.2), (1.1.1b) has the following property:

$$u(s^-(t),t) - u(s^+(t),t) = J(t), \quad 0 \le t \le T. \tag{2.1.55}$$

It follows from the preceding discussion that we should only expect equations (1.1.6) and (1.1.7) to be satisfied by the function $u(x,t) = \lim_{n\uparrow\infty} \{(u^n(x))_{\tau=t/n}\}$, when it exists, in some integrated sense (in time).

If we average equation (1.1.23) over an interval of values of a and b, so that the left hand side becomes

$$\frac{1}{2h}\int_{-h}^{h}\int_{a+\eta}^{b+\eta} f^n(x,t,\alpha)\,dx\,d\eta ,$$

then we will obtain averaged versions of (1.1.6) and (1.1.7) which will hold in the limit $\tau \downarrow 0$, by two applications of the continuity result (2.1.50), with $a + n\tau V$, $b - n\tau V$ replaced by a-h, a+h and b-h, b+h, respectively.

Although we have not discussed the case of the boundary value problem on the finite interval $[x_0, x_1]$ in this section, we may expect, from the discussion following (1.1.33), that the appropriate data to specify at the boundary points x_0 and x_1 are $\tilde{u}^+(x_0,t)$ and $\tilde{u}^-(x_1,t)$ a.u.c. or a.c. in t.

2. Convergence and Error Estimate

The principal result of this section is that, when the velocity v(u) is a monotonically non-increasing or non-decreasing function of u, then the thickness of "shocks" generated by the algorithm (1.1.12)-(1.1.14) is never more than $2V\tau$, where V is given by (1.1.26) and (1.1.28). Furthermore, the shocks so generated are within the distance $2V\tau$ of their position in the exact solution. In order, however, to indicate how one may obtain other types of error estimates which might be more useful in certain situations, and to show why we will want to restrict outselves to monotone v if we are to get particularly sharp error bounds, we shall proceed initially in a somewhat more general vein. Our deliberations will be expedited by the results we obtained in the last section.

So far we have not stated what assumptions we make on the function $v(\cdot)$. But it is clear from (2.1.29) that, if the operator "I" in the algorithm is to be meaningful, then we should at least require v to be measurable. When v is piecewise continuous and $u_0(x)$ is a.u.c., we have already seen, in lemma 2.1.3 and the discussion following it, that we may replace v by a piecewise constant function $\tilde{v}$. The study of convergence of the algorithm (1.1.12)-(1.1.14) for piecewise constant v should be a relatively simple matter, since we may imagine that we are dealing with the propagation of a finite number of individual "streams", each moving with a constant velocity, with a specified law of interaction, and we know from Chapter One that the algorithm gives the exact solution when each of these streams propagates independently.

In fact, when $u_0(x)$ is a.u.c. it should be possible to prove convergence when all we require is that v be bounded measurable. For, by the monotonicity properties shown in the last section,

we may bound u^n above and below by $\tilde{\theta}^n$ and $\tilde{\psi}^n$, respectively, where

$$\tilde{\theta}^o(x) = \sup_{y \in [m\delta,(m+1)\delta)} u_o(y), \quad x \in [m\delta,(m+1)\delta) \ , \tag{2.2.1a}$$

$$\tilde{\theta}^{n+1} = IT_{v\tau}\, D\, \tilde{\theta}^n \ , \tag{2.2.1b}$$

and

$$\tilde{\psi}^o(x) = \inf_{y \in [m\delta,(m+1)\delta)} u_o(y), \quad x \in [m\delta,(m+1)\delta) \ , \tag{2.2.2a}$$

$$\tilde{\psi}^{n+1} = IT_{v\tau}\, D\, \tilde{\psi}^n \ . \tag{2.2.2b}$$

In (2.2.1) and (2.2.2), m is an integer, $-\infty < m < \infty$. Clearly $u_o \le \tilde{\theta}^o \le S_\delta^+ u_o$ and $u_o \ge \tilde{\psi}^o \ge S_\delta^- u_o$. Accordingly, the argument used to prove lemma 2.1.2 can be used, and we find

$$\int_{a+n\tau V}^{b-n\tau V} |\tilde{\theta}^n(x) - u^n(x)|\,dx \le \epsilon(\delta) \ ,$$

$$\int_{a+n\tau V}^{b-n\tau V} |\tilde{\psi}^n(x) - u^n(x)|\,dx \le \epsilon(\delta) \ . \tag{2.2.3}$$

Note that the initial data $\tilde{\theta}^o$ and $\tilde{\psi}^o$ in (2.2.1a) and (2.2.2a) are "histograms". Hence when v is bounded measurable we need only prove convergence for such initial profiles. The exact solutions for such profiles can be given for such v. We know that a "jump" at $x = s(t)$ from u_o to u_1 moves with velocity

$$\frac{ds}{dt} = \frac{F(u_1) - F(u_o)}{u_1 - u_o} \ , \tag{2.2.4}$$

where F is given by (1.1.2b). In fact, when the initial profile is such a histogram involving only a finite number of values of u,

say, $\{u_i, i=1,\cdots,N\}$, it is clear that the exact solution is unchanged by changes in the velocity field v which leave differences of values of F at these values $\{u_i\}$ invariant. What sort of convergence result we may expect for our algorithm for v bounded measurable when the initial profile is a jump, will be indicated in the sequel.

By lemma 2.1.2, we may deal with functions w^n whose peaks and troughs are all of width $\geq 2\delta$. As we pointed out in the discussion accompanying the lemma, this permits us to restrict ourselves to a portion of a profile $w^\rho(x)$ which is monotonically non-decreasing or non-increasing. Without loss of generality, we take the latter case. Given such a portion of a profile, say, for $\xi_0 \leq x \leq \xi_1$, we may let

$$U^- = \inf_{x\in[\xi_0,\xi_1]} w^\rho(x), \quad U^+ = \sup_{x\in[\xi_0,\xi_1]} w^\rho(x) . \tag{2.2.5}$$

Then

$$\lim_{x\downarrow\xi_0} w^\rho(x) = U^+, \quad \lim_{x\uparrow\xi_1} w^\rho(x) = U^- , \tag{2.2.6}$$

and we may imagine the initial profile $w^\rho(x)$ extended so that

$$w^\rho(x) = U^+, \quad x < \xi_0 , \tag{2.2.7a}$$

$$w^\rho(x) = U^-, \quad x > \xi_1 . \tag{2.2.7b}$$

With this profile, we find that

$$w^{n+1} = IT_{v\tau} D w^n , \tag{2.2.8}$$

so that effectively we are considering again the functions generated by the original algorithm (1.1.12)-(1.1.14), or (2.1.29). Thus, we shall now write u^ρ, u^n for w^ρ, w^n.

Before we proceed with a discussion of convergence, let us write down the exact solution of (1.1.2) with initial data $u(x,0) = u^0(x)$. The proof of convergence and error estimate will then be obtained by comparing the approximate solutions $u^n(x)$ with the exact solution $u(x,n\tau)$.

With a monotonically non-increasing profile $g(x)$, $\lim_{x\to-\infty} g(x)=U^+$, $\lim_{x\to+\infty} g(x) = U^-$, we associate a function $X(g;\alpha)$ with range the extended real line for $\alpha \in [U^-,U^+]$, as follows:

$$(Dg)(x,\alpha)=\begin{cases} 1 & x \le X(g;\alpha) \\ 0 & x > X(g;\alpha) \end{cases} \quad . \tag{2.2.9}$$

For the solution of (1.1.2) with $u(x,0) = u^0(x)$, we use the notation

$$X_t(\alpha) \equiv X(u(\cdot,t);\alpha) \quad . \tag{2.2.10}$$

Note that

$$X_t(\alpha) \le X_t(\alpha_1) \quad \text{if} \quad \alpha \ge \alpha_1 \quad . \tag{2.2.11}$$

We will have given $u(x,t)$ if we know $X_t(\alpha)$.

Roughly speaking, the exact solution at t will have degenerated into a sequence of shocks. In fact we may regard it as a continuum of shocks if we regard a continuous profile $g(x)$ as having a jump $d\alpha$ at $X(g;\alpha)$. Still speaking roughly, we first search for the position of the "leftmost" shock, by finding

$$\inf_{\alpha\in[U^-,U^+]} \left\{ \frac{\int_\alpha^{U^+} X_0(\alpha')d\alpha'}{U^+ - \alpha} + \frac{F(U^+)-F(\alpha)}{U^+ - \alpha} t \right\} . \tag{2.2.12}$$

Note that, from (2.2.11),

$$\frac{\int_\alpha^{U^+} X_0(\alpha')d\alpha'}{U^+ - \alpha}$$

is non-increasing in α. Let the infemum in (2.2.12) be achieved for α_0. We then proceed to find the next shock in the continuum of shocks. (Those who object to the "rigor" of the argument may prefer that we start with an initial $u(x,0)$ given as a non-increasing histogram. Then the shocks will be denumerable.) The "next" shock will be located at

$$\inf_{\alpha\in[U^-,\alpha_0]} \left\{ \frac{\int_\alpha^{\alpha_0} X_0(\alpha')d\alpha'}{\alpha_0 - \alpha} + \frac{F(\alpha_0)-F(\alpha)}{\alpha_0 - \alpha} t \right\} .$$

In this way we proceed until we have located all the shocks.

More precisely, given $\epsilon > 0$ and $\beta \in (U^-,U^+]$, let

$$A_\varepsilon(\beta,t) \equiv \left\{ \alpha \middle| \alpha\in[U^-,\beta),\ \alpha - \inf_{\gamma\in[U^-,\beta)} \left(\frac{\int_\gamma^\beta X_0(\alpha')d\alpha'}{\beta - \gamma} + \frac{F(\beta)-F(\gamma)}{\beta - \gamma} t \right) < \varepsilon \right\} , \tag{2.2.13a}$$

$$\bar{A}_\varepsilon(\beta,t) = \text{ closure of } A_\varepsilon(\beta,t) \quad . \tag{2.2.13b}$$

Define

$$\alpha_0(\beta,t) \equiv \inf \{\alpha | \alpha\in\bar{A}_\varepsilon(\beta,t)\ \forall\ \varepsilon > 0\} \quad . \tag{2.2.14}$$

It is clear that $\alpha_0(\beta,t)$ is non-increasing in β. Further, set

$$\alpha^+(\beta,t) \equiv \sup_{\alpha\in[U^-,U^+]} \{\alpha | \beta\in[\alpha_0(\alpha,t),\alpha]\} \tag{2.2.15a}$$

and

$$\alpha^-(\beta,t) \equiv \alpha_0(\alpha^+(\beta,t),t) \quad . \tag{2.2.15b}$$

Then the exact solution of (1.1.2) with $u(x,0) = u^o(x)$ is given, for $\beta \in (U^-,U^+]$, by

$$X_t(\beta) = \lim_{\epsilon \downarrow 0} \left\{ \frac{\int_{\alpha^-(\beta,t)-\epsilon}^{\alpha^+(\beta,t)} X_0(\alpha')\,d\alpha'}{\alpha^+(\beta,t)-\alpha^-(\beta,t)+\epsilon} + \frac{F(\alpha^+(\beta,t)-F(\alpha^-(\beta,t)-\epsilon)}{\alpha^+(\beta,t)-\alpha^-(\beta,t)+\epsilon}\, t \right\} . \tag{2.2.16}$$

Returning to the approximate solutions, we see that, if $g(x) = h(x+\delta)$, then

$$(IT_{v\tau}Dg)(x) = (IT_{v\tau}Dh)(x+\delta) \quad . \tag{2.2.17}$$

Accordingly, if g is monotonically non-increasing and $\delta \geq 0$, so that $g(x) \leq h(x)$, we get from the monotonicity of the operators I, $T_{v\tau}$, D that

$$(IT_{v\tau}Dg)(x) \leq (IT_{v\tau}Dh)(x) = (IT_{v\tau}Dg)(x-\delta) \quad , \tag{2.2.18}$$

and thus the functions $u^n(x)$ will be monotonically non-increasing if u^o is. Given $u^n(x)$, using the notation (2.2.9), we define

$$X_n^*(\alpha) = X(u^n(\cdot);\alpha) \quad . \tag{2.2.19}$$

In addition, we define $x_n^*(\alpha)$, $\alpha \in [U^-,U^+]$, by

$$(T_{v\tau}Du^n)(x,\alpha) = \begin{cases} 1 & x \leq x_n^*(\alpha) \\ 0 & x > x_n^*(\alpha) \end{cases} \quad . \tag{2.2.20}$$

Note that

$$x_n^*(\alpha) = X_n^*(\alpha) + \tau v(\alpha) \tag{2.2.21}$$

and

$$X_n^*(\alpha) \le X_n^*(\alpha_1) \quad \text{if} \quad \alpha \ge \alpha_1 \quad . \tag{2.2.22}$$

It might seem that, if for some n and $\alpha > \alpha_1$ we have $x_n^*(\alpha) > x_n^*(\alpha_1)$ (thereby implying by (2.2.21) and (2.2.22) that $v(\alpha) > v(\alpha_1)$), we would have $|x_m^*(\alpha)-x_m^*(\alpha_1)| \le 2V\tau$ for $m \ge n$. Such a result could then be used to bound the width of a numerical "shock", or rapid change in u including $[\alpha_1,\alpha]$, and lead to a convergence result and error estimate. Our expectation is wrong, however, as we may see by considering the example

$$u^0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \tag{2.2.23a}$$

with the velocity profile

$$v(\alpha) = \begin{cases} 0 & 0 < \alpha < \frac{1}{2} \\ -1-\epsilon & \frac{1}{2} < \alpha < \frac{3}{4} \\ 1 & \frac{3}{4} < \alpha < 1 \end{cases}, \quad \epsilon > 0 \ . \tag{2.2.23b}$$

The exact solution of (1.1.2) with initial condition (2.2.23a) and velocity profile (2.2.23b) is

$$u(x,t) = \begin{cases} 1 & x < -\frac{1}{2}\epsilon t \\ \frac{1}{2} & -\frac{1}{2}\epsilon t < x < 0 \\ 0 & x > 0 \end{cases} \ . \tag{2.2.24}$$

We have not shown that the approximation to (2.2.24) does not have numerical shocks whose width is bounded by a multiple of τ, but only that a conjecture which would readily lead to a bound of that type is invalid. The violation of the conjecture seems to have something to do with the non-monotonicity of $v(\alpha)$ in (2.2.23b).

A more graphic demonstration that we cannot hope to have numerical shocks of thickness $O(\tau)$ as $\tau \downarrow 0$ when v is merely required to be measurable is given by the following example. Suppose that $v(\alpha)$ is measurable and takes on the values 0 and 1, such that the measure of the subset of any interval in which v is 1 is one half the Lebesgue measure of the interval, and the measure of the subset in which v is 0 is also one half the Lebesgue measure of the interval. Consider the initial profile given in (2.2.23a). We know that the exact solution to the problem is

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t \end{cases} . \tag{2.2.25}$$

On the other hand, we compute

$$u^1(x) = \begin{cases} 1 & x < 0 \\ \frac{1}{2} & 0 < x < \tau \\ 0 & x > \tau \end{cases} , \tag{2.2.26a}$$

$$u^2(x) = \begin{cases} 1 & x < 0 \\ \frac{3}{4} & 0 < x < \tau \\ \frac{1}{4} & \tau < x < 2\tau \\ 0 & x > 2\tau \end{cases} , \tag{2.2.26b}$$

$$u^3(x) = \begin{cases} 1 & x < 0 \\ \frac{7}{8} & 0 < x < \tau \\ \frac{1}{2} & \tau < x < 2\tau \\ \frac{1}{8} & 2\tau < x < 3\tau \\ 0 & x > 3\tau \end{cases} , \tag{2.2.26c}$$

$$u^4(x) = \begin{cases} 1 & x < 0 \\ \frac{15}{16} & 0 < x < \tau \\ \frac{11}{16} & \tau < x < 2\tau \\ \frac{5}{16} & 2\tau < x < 3\tau \\ \frac{1}{16} & 3\tau < x < 4\tau \\ 0 & x > 4\tau \end{cases} , \tag{2.2.26d}$$

and so on. It is apparent that we get a binomial distribution for the "front", which becomes a Gaussian distribution in $u^n(x)$ as $n\uparrow\infty$. Accordingly, the width of the front is $O(\sqrt{nV\tau}) = O(\sqrt{t\tau V})$ where $t = n\tau$. In this case the time step τ gives an effective "viscosity" $O(V^2\tau)$.

A natural conjecture to make is that when $v(\cdot)$ is bounded measurable the width of shocks is $O(\sqrt{t\tau V})$. We do nothing further in this report to establish the conjecture, but it would appear to be susceptible to proof by probabilistic methods.

For the remainder of this section we focus on the case where v is monotonic. Since we have already required the initial profile $u^0(x)$ to be non-increasing and we know from section 1.1 that the algorithm (1.1.12)-(1.1.14) gives the exact solution to the problem in this case when v is monotonically non-increasing, we shall consider the other case, where v is monotonically non-decreasing. That is, we require

$$v(\alpha) \geq v(\alpha_1) \quad \text{if} \quad \alpha \geq \alpha_1 \quad . \tag{2.2.27}$$

With the assumption (2.2.27) and the various simplifications of the general initial value problem which have been made, we could plunge directly into a proof of our central result. We will do this shortly, but first we will consider some simpler results which illustrate the ideas we use in carrying out the proof. There will be some further discussion of the types of situations which may arise in implementing the algorithm (1.1.12)-(1.1.14), in order to better motivate the language in which our central result is cast.

We begin by verifying, for the case (2.2.27), the conjecture tentatively put forth after equation (2.2.22), and shown to be invalid for general velocity profiles $v(\alpha)$.

Lemma 2.2.1: If the velocity profile satisfies (2.2.27) and for some $n \geq 0$, $\alpha_1 \geq \alpha_2$,

$$x_n^*(\alpha_1) \geq x_n^*(\alpha_2) \quad , \tag{2.2.28}$$

then for $m \geq n$,

$$\left| x_m^*(\alpha_1) - x_m^*(\alpha_2) \right| \leq 2V\tau \quad . \tag{2.2.29}$$

Proof: We shall show that

$$X_m^*(\alpha_2) - 2V\tau \leq X_m^*(\alpha_1) \leq X_m^*(\alpha_2), \quad m \geq n. \tag{2.2.30}$$

From this we get, by (2.2.21) and (1.1.28),

$$x_m^*(\alpha_2) - 2V\tau \leq x_m^*(\alpha_1) \leq x_m^*(\alpha_2) + 2V\tau \quad ,$$

which is the same as (2.2.29). Equation (2.2.30) can be established for $m = n$, since

$$X_n^*(\alpha_2) - X_n^*(\alpha_1) = x_n^*(\alpha_2) - x_n^*(\alpha_1) + \tau(v(\alpha_1)-v(\alpha_2)) \leq 2V\tau \quad , \tag{2.2.31}$$

by (2.2.28) and (1.1.28). We suppose that (2.2.30) is true for m, and prove it for m + 1.

One half of (2.2.30) is obvious:

$$X_{m+1}^*(\alpha_1) \leq X_{m+1}^*(\alpha_2) \quad .$$

For the other half, recall from the algorithm (1.1.12)-(1.1.14) that

$$X^*_{m+1}(\alpha_2) \le \sup_{\alpha\in[\alpha_2,U^+]} x^*_m(\alpha) , \tag{2.2.32a}$$

$$X^*_{m+1}(\alpha_1) \ge \inf_{\alpha\in[U^-,\alpha_1]} x^*_m(\alpha) . \tag{2.2.32b}$$

So

$$X^*_{m+1}(\alpha_2) - X^*_{m+1}(\alpha_1) \le \sup_{\alpha\in[\alpha_2,U^+]} x^*_m(\alpha) - \inf_{\alpha\in[U^-,\alpha_1]} x^*_m(\alpha)$$

$$= \max\Bigg(\sup_{\alpha\in[\alpha_1,U^+]} x^*_m(\alpha) - \inf_{\alpha\in[U^-,\alpha_1]} x^*_m(\alpha), \quad \sup_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha)$$

$$- \inf_{\alpha\in[U^-,\alpha_2]} x^*_m(\alpha), \quad \sup_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha) - \inf_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha)\Bigg)$$

$$\le \max\left(2V\tau, \quad \sup_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha) - \inf_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha)\right) , \tag{2.2.33}$$

since

$$x^*_m(\alpha) - x^*_m(\beta) \le 2V\tau \quad \text{for} \quad \alpha \ge \beta . \tag{2.2.34}$$

Given $\epsilon_1 > 0$, $\epsilon_2 > 0$, we find α_3 and α_4 such that

$$\begin{aligned} &\alpha_3 \in [\alpha_2,\alpha_1], \quad x^*_m(\alpha_3) \ge \sup_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha) - \epsilon_1 , \\ &\alpha_4 \in [\alpha_2,\alpha_1], \quad x^*_m(\alpha_4) \le \inf_{\alpha\in[\alpha_2,\alpha_1]} x^*_m(\alpha) + \epsilon_2 . \end{aligned} \tag{2.2.35}$$

If $\alpha_3 \geq \alpha_4$, by (2.2.34) and (2.2.33) we get

$$X^*_{m+1}(\alpha_2) - X^*_{m+1}(\alpha_1) \leq 2V\tau + \epsilon_1 + \epsilon_2 \quad . \tag{2.2.36}$$

So suppose $\alpha_3 < \alpha_4$. By (2.2.30),

$$X^*_m(\alpha_3) - 2V\tau \leq X^*_m(\alpha_2) - 2V\tau \leq X^*_m(\alpha_1) \leq X^*_m(\alpha_4) \leq X^*_m(\alpha_3) \ . \tag{2.2.37}$$

Using (2.2.21) and (2.2.27), we find that this leads to

$$x^*_m(\alpha_3) - 2V\tau \leq x^*_m(\alpha_4) \quad ,$$

and once again we get (2.2.36). Since $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are arbitrary, we get

$$X^*_{m+1}(\alpha_2) - X^*_{m+1}(\alpha_1) \leq 2V\tau \ . \tag{2.2.38}$$

This establishes (2.2.30) by induction, and the lemma is proven.

The lemma shows that, if $X^*_n(\alpha_1)$ and $X^*_n(\alpha_2)$ are "close", then so will be $X^*_m(\alpha_1)$ and $X^*_m(\alpha_2)$ for $m \geq n$. We may think of this as giving a bound on the width of numerical shocks. The result is still not as strong as we desire, since we have no idea how many such shocks there may be. One may envisage the algorithm (1.1.12)-(1.1.14) as unfolding through the "interweaving" of various streams. It then appears that, once two such streams are interwoven, they cannot subsequently become unwoven. Thus, we will naturally want to study families of such interwoven streams, in the hope that they will be approximations to the shocks of the exact solution. To see what sorts of families of streams we should consider, we may return to our initial heuristic discussion of the exact solution, where, we recall, we searched for the location of shocks from "left" to "right", minimizing expressions like (2.2.12), and proceeding downward from the "top" U^+ of the velocity profile to the "bottom", at U^-.

Accordingly, we might consider sets of the form

$$B_m \equiv \{\alpha | \alpha \in [U^-, U^+],\ \exists\ \beta \in [U^-, \alpha] \text{ such that } x_n^*(\beta) - x_n^*(U^+) \le 0$$

$$\text{for some } n \le m - 1\} \quad . \tag{2.2.39}$$

If we let

$$\varphi_m \equiv \inf\{v(\alpha) | \alpha \in B_m\} \tag{2.2.40a}$$

and

$$\xi_m \equiv \sup\{X_m^*(\alpha) | \alpha \in B_m\} \ , \tag{2.2.40b}$$

we can prove, by induction in m, that

$$\xi_m - X_m^*(U^+) \le \tau(v(U^+) - \varphi_m) \quad . \tag{2.2.41}$$

However, it may be that a stream becomes interwoven with a second stream which has itself become interwoven directly with the stream U^+. Such streams would not appear directly in the set B_m introduced in (2.2.39). A natural extension of the result given in the last paragraph then involves, given a set $D \subset [U^-, U^+]$, introducing a set

$$B_{X_o^*}(D) \equiv \{\alpha | \alpha \in [U^-, U^+],\ \exists\ \beta \in [U^-, \alpha] \text{ and } \exists\ \delta \in D \text{ such that } x_o^*(\beta) - x_o^*(\delta) \le 0\} . \tag{2.2.42}$$

We observe that

$$B_{X_o^*}(D) \supset D \tag{2.2.43a}$$

and

$$B_{X_o^*}(D) = B_{X_o^*}(D) \tag{2.2.43b}$$

where by $\hat{S}$ we mean the convex hull of a set S:

$$\hat{S} = \{\alpha | \exists\, \beta \in S \text{ and } \exists\, \gamma \in S \text{ such that } \beta \le \alpha \le \gamma\} \quad . \qquad (2.2.44)$$

Then, we can prove the following: If

$$\sup\{X_0^*(\alpha) | \alpha \in D\} - \inf\{X_0^*(\alpha) | \alpha \in D\} \le d \ , \qquad (2.2.45a)$$

then

$$\sup\{X_1^*(\alpha) | \alpha \in B_{X_0^*}(D)\} - \inf\{X_1^*(\alpha) | \alpha \in B_{X_0^*}(D)\}$$

$$\le \max\{d + \tau[\sup\{v(\alpha) | \alpha \in B_{X_0^*}(D) - \hat{D}\} - \inf\{v(\alpha) | \alpha \in B_{X_0^*}(D) - \hat{D}\}],$$

$$\tau[\sup\{v(\alpha) | \alpha \in B_{X_0^*}(D)\} - \inf\{v(\alpha) | \alpha \in B_{X_0^*}(D)\}]\ \} \quad . \qquad (2.2.45b)$$

We are getting progressively larger sets of interwoven streams, but we still have not achieved "closure" of these families. The next natural step is, with $B_{X_0^*}(D)$ defined by (2.2.42), to generate the sequence of sets

$$\begin{aligned} B^{(1)} &= B_{X_0^*}(D) \ , \\ B^{(2)} &= B_{X_0^*}(B^{(1)}) \ , \ \cdots \ , \qquad (2.2.46) \\ B^{(m+1)} &= B_{X_0^*}(B^{(m)}) \ , \ \cdots \ . \end{aligned}$$

Since $B^{(m+1)} \supset B^{(m)}$ and $B^{(m)} \subset [U^-, U^+]\ \forall\ m \ge 1$, we get convergence, and can define

$$\tilde{B}_{X_0^*}(D) \equiv \bigcup_{m=1}^{\infty} B^{(m)} \ . \qquad (2.2.47)$$

Then

$$B_{X_o^*}(\tilde{B}_{X_o^*}(D)) = \tilde{B}_{X_o^*}(D) \quad . \tag{2.2.48}$$

With this much as background, let us finally define the sets which will appear in the statement of our principal result. We have partitions Π_n of the interval $[U^-,U^+]$. By this, we mean that there are ordered sets $A_n \subset [U^-,U^+]$ of points, labeled by the parameter α, such that with each $\alpha \in A_n$ there is associated a set $D_n(\alpha)$ with the following properties:

$$\beta \le \alpha \quad \text{if} \quad \beta \in D_n(\alpha) \ , \tag{2.2.49a}$$

$$\alpha_1 \in A_n,\ \ \alpha_2 \in A_n,\ \ \alpha_1 > \alpha_2,\ \ \beta_1 \in D_n(\alpha_1),\ \ \beta_2 \in D_n(\alpha_2) \Rightarrow \beta_1 > \beta_2 \quad . \tag{2.2.49b}$$

We further require that

$$U^+ \in A_n \ \forall\ n \ge 0 \tag{2.2.50a}$$

and

$$\bigcup_{\alpha \in A_n} D_n(\alpha) = [U^-,U^+] \ \forall\ n \ge 0 \ . \tag{2.2.50b}$$

We may restate (2.2.49b) as

$$\alpha_1 \in A_n,\ \ \alpha_2 \in A_n,\ \ \alpha_1 \ne \alpha_2 \Rightarrow D_n(\alpha_1) \cap D_n(\alpha_2) = \phi \ . \tag{2.2.51}$$

The sets A_n will have the property that

$$A_n \subset A_{n-1} \subset \cdots \subset A_0 = [U^-,U^+] \quad . \tag{2.2.52}$$

Thus,

$$D_0(\alpha) = \{\alpha\}, \quad \alpha \in [U^-, U^+] \quad . \tag{2.2.53}$$

(2.2.53) describes the partition Π_0. We will have described all the partitions if we can show how to get from Π_n to Π_{n+1}.

To get from Π_n to Π_{n+1} we do the following: We form the function $X_n^*(\alpha)$ in accordance with the algorithm (1.1.12)-(1.1.14) and the definition (2.2.19). Suppose we have a set $D \subset [U^-, U^+]$. Given D, we define a number $a_n(D)$:

$$a_n(D) \equiv \sup\{\alpha | \alpha \in A_n, \quad \exists\, \beta \in D \text{ such that } \beta \in D_n(\alpha)\} \; . \tag{2.2.54}$$

Now we define another set by

$$C_{X_n^*}(D) \equiv \{\alpha | \alpha \in [U^-, a_n(D)], \exists\, \gamma \in A_n, \quad \exists\, \beta \in [U^-, \gamma], \text{ and } \exists\, \delta \in D \text{ such that } \alpha \in D_n(\gamma) \text{ and } x_n^*(\beta) - x_n^*(\delta) \le 0\} \; . \tag{2.2.55}$$

Note that

$$C_{X_n^*}(D) \supset D \tag{2.2.56a}$$

and

$$\widehat{C_{X_n^*}(D)} = C_{X_n^*}(D) \quad , \tag{2.2.56b}$$

where we use the notation (2.2.44). Also, if

$$D_n(\alpha) \cap C_{X_n^*}(D) \neq \phi \quad , \tag{2.2.57a}$$

then

$$D_n(\alpha) \subset C_{X_n^*}(D) \quad , \tag{2.2.57b}$$

and, if

$$D_1 \subset D_2 \quad , \tag{2.2.58a}$$

then

$$C_{X_n^*}(D_1) \subset C_{X_n^*}(D_2) \quad . \tag{2.2.58b}$$

Next we generate the sequence

$$\begin{aligned} \Gamma^{(1)}(D) &= C_{X_n^*}(D) \ , \\ \Gamma^{(2)}(D) &= C_{X_n^*}(\Gamma^{(1)}(D)), \cdots \ , \\ \Gamma^{(m+1)}(D) &= C_{X_n^*}(\Gamma^{(m)}(D)), \cdots \ . \end{aligned} \tag{2.2.59}$$

Since $\Gamma^{(m)}(D) \subset [U^-, U^+]$ and $\Gamma^{(m+1)} \supset \Gamma^{(m)} \quad \forall\ m \geq 1$, we can define

$$\tilde{C}_{X_n^*}(D) \equiv \bigcup_{m=1}^{\infty} \Gamma^{(m)}(D) \quad . \tag{2.2.60}$$

For each $\alpha \in A_n$ we find

$$D_{n+1}(\alpha) \equiv \tilde{C}_{X_n^*}(D_n(\alpha)) \quad . \tag{2.2.61}$$

Clearly,

$$C_{X_n^*}(D_{n+1}(\alpha)) = D_{n+1}(\alpha) \quad . \tag{2.2.62}$$

It follows from (2.2.54), (2.2.55), (2.2.59), and (2.2.60) that $U^+ \in D_{n+1}(U^+)$ and $U^+ \notin D_{n+1}(\alpha)$ for $\alpha \in A_n - \{U^+\}$. Now we take a subset of points $\alpha \in A_n$ such that the associated sets $D_{n+1}(\alpha)$ are a disjoint collection of sets covering $[U^-, U^+]$. From the above, we must have U^+ in this subset. Tentatively, we call this subset of

points A_{n+1}. Since we shall see in the next paragraph that A_{n+1} thus defined is determined uniquely, we have also defined the partition Π_{n+1}.

To see that A_{n+1} is defined uniquely, we proceed by induction. It is clear from (2.2.52) that A_0 is uniquely determined. Assume that A_n is. Now, suppose that A_{n+1} is not defined uniquely, so that there are two sets, A_{n+1} and A'_{n+1}, satisfying the conditions of the definition. Let $\alpha \in A_{n+1}$, $\alpha \notin A'_{n+1}$. Since $A_{n+1} \subset A_n$, $\alpha \in A_n$. Also, it is clear from the preceding results that $\alpha \neq U^+$. Then $\exists\, \alpha^{(1)} \in A_n$ such that $\alpha^{(1)} > \alpha$, $\alpha \in D_{n+1}(\alpha^{(1)})$, and $\alpha^{(1)} \in A'_{n+1}$. Since $A'_{n+1} \subset A_n$, $\alpha^{(1)} \in A_n$. If $\alpha^{(1)} \in A_{n+1}$ and $\alpha \in D_{n+1}(\alpha^{(1)})$, we get $\alpha \notin A_{n+1}$, by the requirement that the sets $\{D_{n+1}(\alpha) | \alpha \in A_{n+1}\}$ be disjoint. Since this is a contradiction, we find $\alpha^{(1)} \notin A_{n+1}$. So $\exists\, \alpha^{(2)} > \alpha^{(1)}$ with $\alpha^{(2)} \in A_{n+1} \subset A_n$ and $\alpha^{(1)} \in D_{n+1}(\alpha^{(2)})$. Since $\alpha^{(1)} \in D_{n+1}(\alpha^{(2)}) = C_{X_n}^*(D_{n+1}(\alpha^{(2)}))$ by (2.2.62), it follows from (2.2.57) that $D_n(\alpha^{(1)}) \subset D_{n+1}(\alpha^{(2)})$. From (2.2.58) we get $C_{X_n}^*(D_n(\alpha^{(1)}) \subset D_{n+1}(\alpha^{(2)})$, and in general $\Gamma^{(m)}(D_n(\alpha^{(1)}) \subset D_{n+1}(\alpha^{(2)})$. Hence, from (2.2.60) and (2.2.61), $D_{n+1}(\alpha^{(1)}) \subset D_{n+1}(\alpha^{(2)})$. Since $\alpha \in D_{n+1}(\alpha^{(1)})$, we get $\alpha \in D_{n+1}(\alpha^{(2)})$ for $\alpha \in A_{n+1}$ and $\alpha^{(2)} \in A_{n+1}$. This gives us a contradiction with the required disjointness of the sets $\{D_{n+1}(\alpha) | \alpha \in A_{n+1}\}$. Accordingly, A_{n+1} as defined must be unique.

The sets $D_n(\alpha)$, $\alpha \in A_n$, have the following property: If we replace $u^0(x)$ by

$$u_\gamma^0(x) \equiv \min(u^0(x), \gamma), \quad \gamma \geq \alpha, \tag{2.2.63}$$

then for $0 \leq m \leq n$ $X_m^*(\beta)$ for $\beta \in D_n(\alpha)$ is unchanged.

Let us now prove the following lemma.

Lemma 2.2.2: Let the sets A_n and the sets $D_n(\beta)$, $\beta \in A_n$, be constructed according to the prescription above. Then for $n \geq 0$,

$$\sup\{X_n^*(\alpha)|\alpha \in D_n(\beta)\} - \inf\{X_n^*(\alpha)|\alpha \in D_n(\beta)\} \leq \tau[\sup\{v(\alpha)|\alpha \in D_n(\beta)\} - \inf\{v(\alpha)|\alpha \in D_n(\beta)\}] \quad . \tag{2.2.64}$$

Proof: The case n = 0 is trivial, since, by (2.2.53),

$$\sup\{X_o^*(\alpha)|\alpha \in D_o(\beta)\} = \inf\{X_o^*(\alpha)|\alpha \in D_o(\beta)\} = X_o^*(\beta) \; , \tag{2.2.65a}$$

$$\sup\{v(\alpha)|\alpha \in D_o(\beta)\} = \inf\{v(\alpha)|\alpha \in D_o(\beta)\} = v(\beta) \; . \tag{2.2.65b}$$

We proceed by induction. Thus assume (2.2.64) true for n. We prove it for n + 1.

As in the proof of lemma 2.2.1, we have

$$X_{n+1}^*(\alpha) \leq \sup_{\beta \in [\alpha, U^+]} x_n^*(\beta) \; , \tag{2.2.66a}$$

$$X_{n+1}^*(\alpha) \geq \inf_{\beta \in [U^-, \alpha]} x_n^*(\beta) \; . \tag{2.2.66b}$$

By the construction of $D_{n+1}(\beta)$ with $\beta \in A_{n+1}$, if $\alpha \leq \beta$ and $\alpha \notin D_{n+1}(\beta)$, then $x_n^*(\alpha) > x_n^*(\gamma)$ for all $\gamma \in D_{n+1}(\beta)$. Likewise, if $\alpha \geq \beta$ and $\alpha \notin D_{n+1}(\beta)$, then $x_n^*(\alpha) < x_n^*(\gamma)$ for all $\gamma \in D_{n+1}(\beta)$. Hence the induction will follow for (2.2.64) if we can show that

$$\sup\{x_n^*(\alpha)|\alpha \in D_{n+1}(\beta)\} - \inf\{x_n^*(\alpha)|\alpha \in D_{n+1}(\beta)\}$$
$$\le \tau[\sup\{v(\alpha)|\alpha \in D_{n+1}(\beta)\} - \inf\{v(\alpha)|\alpha \in D_{n+1}(\beta)\}] \ . \tag{2.2.67}$$

First, we try to bound

$$\sup\{x_n^*(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta))\} - \inf\{x_n^*(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta))\} \ . \tag{2.2.68}$$

As in the proof of lemma 2.2.1, this can be approximated arbitrarily well by $x_n^*(\beta_1) - x_n^*(\beta_2)$ for some $\beta_1 \in C_{X_n^*}(D_n(\beta))$ and $\beta_2 \in C_{X_n^*}(D_n(\beta))$. If $\beta_1 \ge \beta_2$,

$$x_n^*(\beta_1) - x_n^*(\beta_2) = X_n^*(\beta_1) - X_n^*(\beta_2) + \tau(v(\beta_1)-v(\beta_2))$$
$$\le (v(\beta_1)-v(\beta_2)) \ , \tag{2.2.69}$$

and we get the desired bound.

Suppose $\beta_1 < \beta_2$. We can write $\beta_1 \in D_n(\gamma_1)$, $\beta_2 \in D_n(\gamma_2)$, such that $\exists\, \beta_1' \le \gamma_1$ and $\beta_2' \le \gamma_2$ with $x_n^*(\beta_1') \le x_n^*(\delta_1)$, $x_n^*(\beta_2') \le x_n^*(\delta_2)$ and $\delta_1 \in D_n(\beta)$, $\delta_2 \in D_n(\beta)$, Clearly, $\beta_1' \in C_{X_n^*}(D_n(\beta))$ and $\beta_2' \in C_{X_n^*}(D_n(\beta))$. Since $\beta_1 < \beta_2$, $\beta_1 \in D_n(\beta) \Rightarrow \beta_2 \in D_n(\beta)$. In that case,

$$x_n^*(\beta_1) - x_n^*(\beta_2) \le X_n^*(\beta_1) - X_n^*(\beta_2) \le \sup\{X_n^*(\alpha)|\alpha \in D_n(\beta)\}$$
$$- \inf\{X_n^*(\alpha)|\alpha \in D_n(\beta)\} \le \tau[\sup\{v(\alpha)|\alpha \in D_n(\beta)\} - \inf\{v(\alpha)|\alpha \in D_n(\beta)\}] \ , \tag{2.2.70}$$

on use of (2.2.64).

So consider the case $\beta_2 \in D_n(\beta)$, $\beta_1 \notin D_n(\beta)$. Suppose $\beta_1' \le \beta_1$, where β_1' is given above. Then

$$x_n^*(\beta_1) - x_n^*(\beta_1') \le \tau(v(\beta_1)-v(\beta_1'))$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))-D_n(\beta)\} - \inf\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))-D_n(\beta)\}]. \quad (2.2.71)$$

Otherwise $\beta_1' > \beta_1$. Since $\beta_1 \in D_n(\gamma_1)$, we have $\beta_1' \in D_n(\gamma_1)$.

$$x_n^*(\beta_1) - x_n^*(\beta_1') \le \sup\{x_n^*(\alpha)|\alpha\in D_n(\gamma_1)\} - \inf\{x_n^*(\alpha)|\alpha\in D_n(\gamma_1)\}, \quad \gamma_1 < \beta. \quad (2.2.72)$$

As we have done before, we approximate the right hand side of (2.2.72) by

$$x_n^*(\alpha_1) - x_n^*(\alpha_2) = X_n^*(\alpha_1) - X_n^*(\alpha_2)$$

$$+ \tau[v(\alpha_1)-v(\alpha_2)], \quad \alpha_1\in D_n(\gamma_1), \quad \alpha_2\in D_n(\gamma_1) \quad . \quad (2.2.73)$$

When $\alpha_1 \le \alpha_2$, an upper bound is

$$X_n^*(\alpha_1) - X_n^*(\alpha_2) \le \tau[\sup\{v(\alpha)|\alpha\in D_n(\gamma_1)\} - \inf\{v(\alpha)|\alpha\in D_n(\gamma_1)\}], \quad (2.2.74)$$

by (2.2.64). When $\alpha_1 > \alpha_2$, we get the same bound directly from (2.2.73). In either case, we get

$$x_n^*(\alpha_1) - x_n^*(\alpha_2)$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))-D_n(\beta)\} - \inf\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))-D_n(\beta)\}],$$

since $D_n(\gamma_1) \subset C_{X_n^*}(D_n(\beta))$ and $D_n(\gamma_1) \cap D_n(\beta) = \phi$. This last bound is the same as the right hand side of (2.2.71), which holds, therefore, whether $\beta_1' \le \beta_1$ or $\beta_1' > \beta_1$.

Since $\beta_2 \in D_n(\beta)$, we get, with δ_1 given as above,

$$x_n^*(\beta_1) - x_n^*(\beta_2) = x_n^*(\beta_1) - x_n^*(\beta_1') + x_n^*(\beta_1') - x_n^*(\delta_1) + x_n^*(\delta_1) - x_n^*(\beta_2)$$

$$\leq \tau[\sup\{v(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta)) - D_n(\beta)\} - \inf\{v(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta)) - D_n(\beta)\}]$$

$$+ \tau[\sup\{v(\alpha)|\alpha \in D_n(\beta)\} - \inf\{v(\alpha)|\alpha \in D_n(\beta)\}]$$

$$\leq \tau[\sup\{v(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta))\}] \quad . \tag{2.2.75}$$

Here we have bounded $x_n^*(\delta_1) - x_n^*(\beta_2)$ in the same way we bounded $x_n^*(\alpha_1) - x_n^*(\alpha_2)$ in (2.2.73).

We have treated the cases $\beta_1 \geq \beta_2$ and $\beta_1 < \beta_2$, $\beta_2 \in D_n(\beta)$. Suppose now $\beta_1 < \beta_2$, $\beta_2 \notin D_n(\beta)$. Consider the case $\beta_2 \leq \gamma_1$, with γ_1 defined above. Then since $\beta_1 < \beta_2$ and $\beta_1 \in D_n(\gamma_1)$, we have $\beta_2 \in D_n(\gamma_1)$. As above,

$$x_n^*(\beta_1) - x_n^*(\beta_2) \leq \tau[\sup\{v(\alpha)|\alpha \in D_n(\gamma_1)\} - \inf\{v(\alpha)|\alpha \in D_n(\gamma_1)\}]. \tag{2.2.76}$$

Otherwise $\beta_2 > \gamma_1$ and $\beta_2 \in D_n(\gamma_2)$, $\gamma_1 < \gamma_2 < \beta$. Choose $\alpha \in D_n(\beta)$ such that $v(\alpha)$ approximates $\inf\{v(\alpha)|\alpha \in D_n(\beta)\}$. Then $\alpha > \beta_2$. So

$$x_n^*(\alpha) - x_n^*(\beta_2) = X_n^*(\alpha) - X_n^*(\beta_2) + \tau[v(\alpha) - v(\beta_2)] \leq \tau[v(\alpha) - v(\beta_2)] \; . \tag{2.2.77}$$

With β_1' given above, we have, for $\beta_1' \leq \beta_1$,

$$x_n^*(\beta_1) - x_n^*(\beta_1') \leq \tau[v(\beta_1) - v(\beta_1')]$$

$$\leq \tau[\sup\{v(\alpha)|\alpha \in D_n(\gamma_1)\} - \inf\{v(\alpha)|\alpha \in C_{X_n^*}(D_n(\beta))\}] \; . \tag{2.2.78}$$

For $\beta_1' > \beta_1$, $x_n^*(\beta_1) - x_n^*(\beta_1')$ can be bounded just as we did in (2.2.72)-(2.2.74). A suitable bound was given by the right hand side of (2.2.74), which in turn can be bounded by the right hand side of (2.2.78). Thus (2.2.78) holds whether $\beta_1' \le \beta_1$ or $\beta_1' > \beta_1$.

Choosing $\alpha \in D_n(\beta)$ appropriately, we can bound $x_n^*(\alpha) - x_n^*(\beta_2)$ above as closely as we desire by $\tau[\inf\{v(\alpha)|\alpha\in D_n(\beta)\} - v(\beta_2)]$. Finally, write

$$x_n^*(\beta_1) - x_n^*(\beta_2) = x_n^*(\beta_1) - x_n^*(\beta_1') + x_n^*(\beta_1') - x_n^*(\delta_1) + x_n^*(\delta_1) - x_n^*(\alpha) + x_n^*(\alpha) - x_n^*(\beta_2)$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in D_n(\gamma_1)\} - \inf\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\}]$$

$$+ \tau[\sup\{v(\alpha)|\alpha\in D_n(\beta)\} - \inf\{v(\alpha)|\alpha\in D_n(\beta)\}]$$

$$+ \tau[\inf\{v(\alpha)|\alpha\in D_n(\beta)\} - v(\beta_2)]$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\}] .$$

Here we have used the fact that $\beta_2 > \gamma_1$, and we have bounded $x_n^*(\delta_1) - x_n^*(\alpha)$ in a now familiar way.

Thus, we obtain

$$\begin{aligned} &\sup\{x_n^*(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\} - \inf\{x_n^*(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\} \\ &\le \tau[\sup\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha\in C_{X_n^*}(D_n(\beta))\}] . \end{aligned} \tag{2.2.79}$$

Now we turn to

$$\sup\{x_n^*(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\} - \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\} , \tag{2.2.80}$$

where $\Gamma^{(2)}(D_n(\beta))$ was given in (2.2.59). As before, this can be approximated by

$$x_n^*(\beta_1) - \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\}, \quad \beta_1\in \Gamma^{(2)}(D_n(\beta)) \quad .$$

For $\alpha \in \Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))$, $x_n^*(\alpha) > x_n^*(\delta)$ for all $\delta \in D_n(\beta)$. Thus,

$$\inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\} = \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\}. \qquad (2.2.81)$$

Suppose $\beta_1 \in \Gamma^{(1)}(D_n(\beta))$. Then $\sup\{x_n^*(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\}$ can be approximated by $\sup\{x_n^*(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\}$.

Consider the other case, $\beta_1 \notin \Gamma^{(1)}(D_n(\beta))$. $\beta_1 \in D_n(\gamma_1)$. Then $\exists\ \beta_1' \le \gamma_1$ such that $x_n^*(\beta_1') \le x_n^*(\delta_1)$ for some $\delta_1 \in \Gamma^{(1)}(D_n(\beta))$. $\beta_1' \in \Gamma^{(2)}(D_n(\beta))$. If $\gamma_1 \in \Gamma^{(1)}(D_n(\beta))$, we get $\beta_1 \in \Gamma^{(1)}(D_n(\beta))$, which is a contradiction. Thus $\gamma_1 \notin \Gamma^{(1)}(D_n(\beta))$. On the other hand,

$\beta_1\in \Gamma^{(2)}(D_n(\beta))$, $\beta_1\in D_n(\gamma_1) \Rightarrow \gamma_1\in \Gamma^{(2)}(D_n(\beta))$. So $\gamma_1\in \Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))$. Similarly, $\beta_1' \le \gamma_1$, $\gamma_1\in \Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))$, and the connectedness of $\Gamma^{(1)}(D_n(\beta))$ imply that $\beta_1' \in \Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))$.

Consider the case $\beta_1' \le \beta_1$.

$$\begin{aligned} x_n^*(\beta_1) - x_n^*(\beta_1') &= X_n^*(\beta_1) - X_n^*(\beta_1') + \tau(v(\beta_1)-v(\beta_1')) \le \tau[v(\beta_1)-v(\beta_1')] \\ &\le \tau[\sup\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))\} \qquad (2.2.82) \\ &\quad - \inf\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))\}] \quad . \end{aligned}$$

In the other case, $\beta_1 < \beta_1'$. Since $\beta_1 \in D_n(\gamma_1)$ we have $\beta_1' \in D_n(\gamma_1)$. As before, we get in this case

$$x_n^*(\beta_1) - x_n^*(\beta_1') \le \tau[\sup\{v(\alpha)|\alpha\in D_n(\gamma_1)\} - \inf\{v(\alpha)|\alpha\in D_n(\gamma_1)\}] ,$$

which in turn is bounded by the right hand side of (2.2.82) since $\gamma_1 \in \Gamma^{(2)}(D_n(\beta)) - \Gamma^{(1)}(D_n(\beta))$.

Thus,

$$\begin{aligned} x_n^*(\beta_1) - \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\} &= x_n^*(\beta_1) - x_n^*(\beta_1') + x_n^*(\beta_1') - x_n^*(\delta_1) \\ &\quad + x_n^*(\delta_1) - \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\} \end{aligned}$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))-\Gamma^{(1)}(D_n(\beta))\}-\inf\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))-\Gamma^{(1)}(D_n(\beta))\}]$$

$$+ \tau[\sup\{v(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha\in\Gamma^{(1)}(D_n(\beta))\}]$$

$$\le \tau[\sup\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha\in\Gamma^{(2)}(D_n(\beta))\}] . \quad (2.2.83)$$

We may continue in this way, getting

$$\begin{aligned} &\sup\{x_n^*(\alpha)|\alpha\in\Gamma^{(m)}(D_n(\beta))\} - \inf\{x_n^*(\alpha)|\alpha\in\Gamma^{(m)}(D_n(\beta))\} \\ &\qquad \le \tau[\sup\{v(\alpha)|\alpha\in\Gamma^{(m)}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha\in\Gamma^{(m)}(D_n(\beta))\}] \end{aligned} \quad (2.2.84)$$

and

$$\begin{aligned} &\sup\{x_n^*(\alpha)|\alpha\in\tilde{C}_{X_n^*}(D_n(\beta))\} - \inf\{x_n^*(\alpha)|\alpha \in \tilde{C}_{X_n^*}(D_n(\beta))\} \\ &\qquad \le \tau[\sup\{v(\alpha)|\alpha\in \tilde{C}_{X_n^*}(D_n(\beta))\} - \inf\{v(\alpha)|\alpha \in \tilde{C}_{X_n^*}(D_n(\beta))\}] . \end{aligned} \quad (2.2.85)$$

This establishes (2.2.67) by induction, and hence (2.2.64) and the lemma.

Thus, for v monotone, the total width of all shocks as we run down from a peak at U^+ to a minimum at U^- is no more than $\tau|v(U^+)-v(U^-)|$. A natural conjecture to make is the following: The total width of all shocks as we run down from a maximum U^+ to a minimum U^- is bounded by

$$\tau \, \mathrm{var}(v;[U^-,U^+]) \; , \tag{2.2.86a}$$

where the variation in the interval is defined by (Reference 8)

$$\mathrm{var}(v;[U^-,U^+]) = \sup_{\Pi} \sum_{i=1}^{n} |v(\alpha_i)-v(\alpha_{i-1})| \tag{2.2.86b}$$

and Π is a partition of $[U^-,U^+]$ as follows:

$$U^- = \alpha_0 < \alpha_1 < \cdots < \alpha_n = U^+ \; . \tag{2.2.86c}$$

We leave the proof (or disproof) of this conjecture to the reader. Since a function of bounded variation is differentiable almost everywhere, it is clear anyway that v of bounded variation in $[U^-,U^+]$ can be approximated as closely as we desire (in the maximum norm) by a piecewise constant v. And for a.u.c. initial data, we have bounded the error induced by such an approximation in lemma 2.1.3.

The situation is this, then: After n time steps we have a profile $X_n^*(\alpha)$ and, for $\beta \in A_n$, shocks spanning the range of velocities in the $D_n(\beta)$. To complete our analysis, we should compare our approximate solutions with the exact solution $X_{n\tau}(\beta)$ given in (2.2.16).

Because of the way the sets $D_n(\beta)$ for $\beta \in A_n$ were constructed, we see that in the first n steps the algorithm (1.1.12)-(1.1.14) produces no "cascading" of the quantity U_1 (cf (1.1.3)) into the sets $D_n(\beta)$ from above, nor out of these sets below. A more precise statement of this property, in terms of the set function U_1 introduced in equation (1.1.39a) and the operator D, is

$$U_1(Du^m;\Omega) = U_1(Du^o;\Omega) + m\tau \int_{D_n(\beta)} v(\alpha)d\alpha, \quad 0 \le m \le n , \tag{2.2.87a}$$

where

$$\Omega \equiv \{(x,\alpha) \mid x \ge X_o^*(U^+) - n\tau V, \ \alpha \in D_n(\beta), \ \beta \in A_n\} . \tag{2.2.87b}$$

Defining, for $\beta \in A_n$,

$$D_<(\beta) \equiv \inf\{\alpha \mid \alpha \in D_n(\beta)\} \tag{2.2.88}$$

we find from (1.1.24) that

$$n\tau \int_{D_<}^{\beta} v(\alpha)d\alpha = \int_{D_<}^{\beta} (X^*_n(\alpha) - X_o^*(\alpha))d\alpha. \tag{2.2.89}$$

Using the result in lemma 2.2.2 that, for $\alpha \in D_n(\beta)$, $\beta \in A_n$,

$$\sup\{X_n^*(\alpha) \mid \alpha \in D_n(\beta)\} - \tau[\sup\{v(\alpha) \mid \alpha \in D_n(\beta)\} - \inf\{v(\alpha) \mid \alpha \in D_n(\beta)\}]$$

$$\le X_n^*(\alpha) \le \inf\{X_n^*(\alpha) \mid \alpha \in D_n(\beta)\} + \tau[\sup\{v(\alpha) \mid \alpha \in D_n(\beta)\} - \inf\{v(\alpha) \mid \alpha \in D_n(\beta)\}] ,$$

we see that (2.2.88) leads to

$$\left| X_n^*(\alpha) - \frac{1}{\beta - D_<} \int_{D_<}^{\beta} X_o^*(\alpha)d\alpha - \frac{n\tau}{\beta - D_<} \int_{D_<}^{\beta} (\alpha)d\alpha \right|$$

$$\le \tau[\sup\{v(\alpha) \mid \alpha \in D_n(\beta)\} - \inf\{v(\alpha) \mid \alpha \in D_n(\beta)\}] , \tag{2.2.90}$$

for all $\beta \in A_n$ such that $D_<(\beta) \ne \beta$. In this paragraph and the following ones, one may, if he desires, consider that the monotonically non-increasing function $u^o(x)$ has been replaced by a histogram, as described in (2.2.1)-(2.2.4). In that case, the sets A_n already introduced and the sets B_t to be introduced are all finite and $D_<(\beta) < \beta$ for $\beta \in A_n$. Our final error estimate will be independent

of the numbers of points in the sets A_n and B_t, and hence independent of how closely we approximate $u^o(x)$ by a histogram, and will hold quite generally. Alternatively, we may use the following simple bound for the error induced in the frontal locations X_t and X_n^* by replacing $u^o(x)$ by the histogram (2.2.1a), for $u^o(x)$ monotonically non-increasing and satisfying the conditions (2.2.5)-(2.2.7). For, it is clear in this case that

$$\tilde{\theta}^o(x) = u_o(m\delta) \le u_o(x-\delta), \quad m\delta \le x < (m+1)\delta \quad . \tag{2.2.91}$$

Since replacing $u_o(x)$ by $u_o(x-\delta)$ will have the effect of replacing X_t and X_n^* by $X_t+\delta$ and $X_n^*+\delta$, respectively, and since the fronts X_t and X_n^* depend monotonically on the initial data, as described in the last section, we see that

$$X_t(u^o) \le X_t(\tilde{\theta}^o) \le X_t(u^o) + \delta \ , \tag{2.2.92a}$$

$$X_n^*(u^o) \le X_n^*(\tilde{\theta}^o) \le X_n^*(u^o) + \delta \ , \tag{2.2.92b}$$

and

$$|X_n^*(u^o)-X_t(u^o)| \le |X_n^*(\tilde{\theta}_o)-X_t(\tilde{\theta}_o)| + \delta \quad . \tag{2.2.92c)}$$

Given the exact solution $X_t(\beta)$ in (2.2.16), we can effect a partition θ_t of $[U^-,U^+]$ as follows: For each $\gamma \in [U^-,U^+]$ we define a set

$$E_t(\gamma) \equiv \{\alpha | \alpha \in [U^-,\gamma], \ X_t(\alpha) = X_t(\gamma)\} \quad . \tag{2.2.93}$$

Then we find a set $B_t \subset [U^-,U^+]$ such that for $\gamma \in B_t$ the sets $\{E_t(\gamma)\}$ are a disjoint covering of $[U^-,U^+]$. We let $B_o = [U^-,U^+]$ and $E_o(\gamma) = \{\gamma\}$. Just as we proved that the sets A_n and the partitions Π_n were defined uniquely, we can show that the B_t and θ_t are uniquely determined. As we defined $D_<(\beta)$ in (2.2.88), we can define, for $\gamma \in B_t$,

$$E_<(\gamma) \equiv \inf\{\alpha | \alpha \in E_t(\gamma)\} \ . \tag{2.2.94}$$

The analogue of (2.2.87) in this case is

$$U_1(Du(\cdot,t');\Omega) = U_1(Du(\cdot,0);\Omega) + t'\int_{E_t(\gamma)} v(\alpha)d\alpha, \quad 0 \le t' \le t, \tag{2.2.95a}$$

where

$$\Omega \equiv \{(x,\alpha) \mid x \ge X_0(U^+) - tV, \ \ \alpha \in E_t(\gamma), \ \ \gamma \in B_t\} . \tag{2.2.95b}$$

Lemma 2.2.3: If $u^0(x) = u(x,0)$,

$$A_n \subset B_{n\tau} . \tag{2.2.96}$$

Proof: It follows from (2.2.10) and (2.2.19) that $X_0^*(\alpha) = X_0(\alpha)$. The lemma is trivially true for $n = 0$. We will prove the lemma for $n \ge 1$ by contradiction. So suppose $\beta \in A_n$ and $\beta \in E_{n\tau}(\gamma)$, $\gamma > \beta$, $\gamma \in B_{n\tau}$. First, consider the case $E_<(\gamma) < \beta$. Over the set

$$\Omega_< \equiv \{(x,\alpha) \mid x \ge X_0(U^+) - n\tau V, \ \ E_<(\gamma) < \alpha < \beta\} , \tag{2.2.97}$$

we can have no gain of $U_1(\Omega)$ in the approximate solution, and we can have no loss of $U_1(\Omega)$ in the exact solution, beyond that which flows across the boundary at $x = X_0(U^+) - n\tau V$. Thus,

$$(\beta - E_<(\gamma) X_{n\tau}(\gamma) \ge \int_{E_<(\gamma)}^{\beta} [X_0(\alpha) + n\tau v(\alpha)]d\alpha , \tag{2.2.98a}$$

$$\int_{E_<(\gamma)}^{\beta} X_n^*(\alpha)d\alpha \le \int_{E_<(\gamma)}^{\beta} [X_0(\alpha) + n\tau v(\alpha)]d\alpha , \tag{2.2.98b}$$

and

$$X_n^*(\beta^-) \le X_{n\tau}(\gamma) . \tag{2.2.98c}$$

Similarly, over the set

$$\Omega_{>} \equiv \{(x,\alpha) \mid x \geq X_0(U^+) - n\tau V, \quad \beta < \alpha < \gamma\} , \tag{2.2.99}$$

we can have no loss of $U_1(\Omega)$ in the approximate solution nor gain of $U_1(\Omega)$ in the exact solution, except for the flows across $x = X_0(U^+) - n\tau V$. This gives

$$\int_\beta^\gamma X_n^*(\alpha)\,d\alpha \geq \int_\beta^\gamma [X_0(\alpha) + n\tau v(\alpha)]\,d\alpha , \tag{2.2.100a}$$

$$(\gamma-\beta)X_{n\tau}(\gamma) \leq \int_\beta^\gamma [X_0(\alpha) + n\tau v(\alpha)]\,d\alpha , \tag{2.2.100b}$$

and

$$X_n^*(\beta^+) \geq X_{n\tau}(\gamma) . \tag{2.2.100c}$$

Since monotonicity of X_n^* requires $X_n^*(\beta^+) \leq X_n^*(\beta^-)$, the inequalities in (2.2.98) and (2.2.100) must be equalities and $X_n^*(\beta) = X_{n\tau}(\gamma)$. In particular, from (2.2.100a,b) and the monotonicity of X_n^* we conclude that

$$X_n^*(\gamma^-) = X_{n\tau}(\gamma) = X_n^*(\beta) .$$

But this violates the definition of $\beta \in A_n$, and we get a contradiction.

Next, consider the case $E_<(\gamma) = \beta$. The results (2.2.100) still hold. On the other hand, from (2.2.16) and our construction (1.1.12)-(1.1.14), we still must have

$$X_{n\tau}(\gamma) = X_{n\tau}(\beta) \geq X_0(\beta) + n\tau v(\beta) = X_0^*(\beta) + n\tau v(\beta) \geq X_n^*(\beta) \tag{2.2.101}$$

if $\beta \in A_n$, the inequalities in (2.2.100) must become equalities, and we get a contradiction as before.

Thus, if $\beta \in A_n$, we also have $\beta \in B_{n\tau}$. Two cases may arise: (i) $D_n(\beta) = E_{n\tau}(\beta)$; or (ii) $\exists\ \gamma \in D_n(\beta)$ with $\gamma < \beta$ and

$\gamma \in B_{n\tau}$. In the first case, the same considerations as above show that

$$\int_{D_<(\beta)}^{\beta} X_n^*(\alpha)d\alpha = (\beta - D_<(\beta))\, X_{n\tau}(\beta) \quad \text{if} \quad D_<(\beta) < \beta \ , \tag{2.2.102a}$$

$$X_n^*(\beta) = X_{n\tau}(\beta) \quad \text{if} \quad D_<(\beta) = \beta \ . \tag{2.2.102b}$$

Either way,

$$\inf\{X_n^*(\alpha) \,|\, \alpha \in D_n(\beta)\} \le X_{n\tau}(\beta) \le \sup\{X_n^*(\alpha) \,|\, \alpha \in D_n(\beta)\} \ . \tag{2.2.103}$$

In the second case, if $D_<(\beta) < \gamma < \beta$, our arguments based on the flow of U_1 indicate that

$$\int_{\gamma}^{\beta} X_n^*(\alpha)d\alpha \le \int_{\gamma}^{\beta} X_{n\tau}(\alpha)d\alpha \tag{2.2.104a}$$

and

$$\int_{D_<(\beta)}^{\gamma} X_n^*(\alpha)d\alpha \ge \int_{D_<(\beta)}^{\gamma} X_{n\tau}(\alpha)d\alpha \ . \tag{2.2.104b}$$

From (2.2.104a) and the monotonicity of both X_n^* and $X_{n\tau}$, we conclude

$$\inf\{X_n^*(\alpha) \,|\, \alpha \in D_n(\beta)\} \le X_{n\tau}(\gamma) \ . \tag{2.2.105a}$$

Likewise, (2.2.104b) leads to

$$X_{n\tau}(\gamma) \le \sup\{X_n^*(\alpha) \,|\, \alpha \in D_n(\beta)\} \ . \tag{2.2.105b}$$

Finally, suppose $\gamma = D_<(\beta) < \beta$. The result (2.2.105a) still holds. On the other hand, a direct check of (2.2.16) and the construction (1.1.12)-(1.1.14) shows that

$$X_{n\tau}(\gamma) = X_o(\gamma) + n\tau v(\gamma) = X_o^*(\gamma) + n\tau v(\gamma) \leq X_n^*(\gamma) \quad . \tag{2.2.106}$$

In all cases considered, we have shown that, for $\gamma \in D_n(\beta)$,

$$\inf\{X_n^*(\alpha) | \alpha \in D_n(\beta)\} \leq X_{n\tau}(\gamma) \leq \sup\{X_n^*(\alpha) | \alpha \in D_n(\beta)\} \quad . \tag{2.2.107}$$

Combining (2.2.107) with (2.2.64), we see that we have proven the following theorem.

Theorem: For $\gamma \in D_n(\beta)$,

$$|X_n^*(\gamma) - X_{n\tau}(\gamma)| \leq \tau[\sup\{v(\alpha) | \alpha \in D_n(\beta)\} - \inf\{v(\alpha) | \alpha \in D_n(\beta)\}] \quad . \tag{2.2.108}$$

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